Online Appendix to:
Socially-Optimal Design of Service Exchange Platforms with Imperfect Monitoring

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This online appendix contains the proof of Propostion 4.1, the proof of Theorem 5.2, and a complete description of the algorithm.

A. PROOF OF PROPOSITION 4.1

A.1. The Claim to Prove

In order to prove Proposition 4.1, we quantify the performance loss of strategies restricted to A^{as}. The performance loss is determined in the following claim.

Claim. Starting from any initial rating profile $\boldsymbol{\theta}$, the maximum social welfare achievable at the PAE by $(\pi_0, \pi \cdot \mathbf{1}_N) \in \Pi(A^{\text{as}}) \times \Pi^N(A^{\text{as}})$ is at most

$$b - c - c \cdot \rho(\boldsymbol{\theta}, \alpha_0^*, S_B^*) \sum_{\boldsymbol{s}' \in S_B^*} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0^*, \alpha^{\text{a}} \cdot \mathbf{1}_N), \tag{29}$$

where α_0^*, the optimal recommended plan, and S_B^*, the optimal subset of rating distributions, are the solutions to the following optimization problem:

$$\begin{aligned} &\min_{\alpha_0} \min_{S_B \subset S} \left\{ \rho(\boldsymbol{\theta}, \alpha_0, S_B) \sum_{\boldsymbol{s}' \in S_B} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha^{\text{a}} \cdot \mathbf{1}_N) \right\} \qquad (30) \\ &s.t. \quad \sum_{\boldsymbol{s}' \in S \setminus S_B} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha^{\text{a}} \cdot \mathbf{1}_N) > \sum_{\boldsymbol{s}' \in S \setminus S_B} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha_i = \alpha^0, \alpha^{\text{a}} \cdot \mathbf{1}_N), \forall i \in \mathcal{N}, \\ &\qquad \sum_{\boldsymbol{s}' \in S \setminus S_B} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha^{\text{a}} \cdot \mathbf{1}_N) > \sum_{\boldsymbol{s}' \in S \setminus S_B} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha_i = \alpha^1, \alpha^{\text{a}} \cdot \mathbf{1}_N), \ \forall i \in \mathcal{N}, \\ &\qquad \sum_{\boldsymbol{s}' \in S \setminus S_B} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha^{\text{a}} \cdot \mathbf{1}_N) > \sum_{\boldsymbol{s}' \in S \setminus S_B} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha_i = \alpha^{01}, \alpha^{\text{a}} \cdot \mathbf{1}_N), \ \forall i \in \mathcal{N}, \end{aligned}$$

where $\rho(\boldsymbol{\theta}, \alpha_0, S_B)$ is defined as

$$\begin{aligned} \rho(\boldsymbol{\theta}, \alpha_0, S_B) \triangleq \max_{i \in \mathcal{N}} \max \Bigg\{ & \frac{\frac{s_{\theta_i} - 1}{N-1}}{\sum_{\boldsymbol{s}' \in S \setminus S_B} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha^{\text{a}} \cdot \mathbf{1}_N) - q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha_i = \alpha^0, \alpha^{\text{a}} \cdot \mathbf{1}_N)}, \\ & \frac{\frac{s_{1-\theta_i}}{N-1}}{\sum_{\boldsymbol{s}' \in S \setminus S_B} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha^{\text{a}} \cdot \mathbf{1}_N) - q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha_i = \alpha^1, \alpha^{\text{a}} \cdot \mathbf{1}_N)}, \\ & \frac{1}{\sum_{\boldsymbol{s}' \in S \setminus S_B} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha^{\text{a}} \cdot \mathbf{1}_N) - q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha_i = \alpha^{01}, \alpha^{\text{a}} \cdot \mathbf{1}_N)} \Bigg\}, \end{aligned} \tag{31}$$

where α^0 (resp. α^1) is the plan in which the user does not serve rating-0 (resp. rating-1) users, and α^{01} is the plan in which the user does not serve anyone.

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This claim shows that

$$W(\varepsilon, \delta, A^{\mathrm{as}}) \leq b - c - c \cdot \rho(\boldsymbol{\theta}, \alpha_0^*, S_B^*) \sum_{\boldsymbol{s}' \in S_B^*} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0^*, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N),$$

for any δ. By defining

$$\zeta(\varepsilon) \triangleq c \cdot \rho(\boldsymbol{\theta}, \alpha_0^*, S_B^*) \sum_{\boldsymbol{s}' \in S_B^*} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0^*, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N),$$

we obtain the result in Proposition 1, namely, $\lim_{\delta \to 1} W(\varepsilon, \delta, A^{\mathrm{as}}) \leq b - c - \zeta(\varepsilon)$. Note that $\zeta(\varepsilon)$ is indeed a function of the rating update error ε, because ε determines the state transition function $q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0^*, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N)$ and thus affects $\rho(\boldsymbol{\theta}, \alpha_0, S_B)$. Note also that $\zeta(\varepsilon)$ is independent of the discount factor δ.

In the expression of $\zeta(\varepsilon)$, $\rho(\boldsymbol{\theta}, \alpha_0, S_B)$ represents the normalized benefit from deviation (normalized by $b-c$). The numerator of $\rho(\boldsymbol{\theta}, \alpha_0, S_B)$ is the probability of a player matched to the type of clients whom it deviates to not serve. The higher this probability, the larger benefit from deviation a player can get. The denominator of $\rho(\boldsymbol{\theta}, \alpha_0, S_B)$ is the difference between the two state transition probabilities when the player does and does not deviate, respectively. When these two transition probabilities are closer, it is less likely to detect the deviation, which results in a larger $\rho(\boldsymbol{\theta}, \alpha_0, S_B)$. Hence, we can expect that a larger $\rho(\boldsymbol{\theta}, \alpha_0, S_B)$ (i.e., a larger benefit from deviation) will result in a larger performance loss, which is indeed true as will be proved later.

We can also see that $\zeta(\varepsilon) > 0$, as long as $\varepsilon > 0$. The reason is as follows. Suppose that $\varepsilon > 0$. First, from (31), we know that $\rho(\boldsymbol{\theta}, \alpha_0, S_B) > 0$ for any $\boldsymbol{\theta}$, α, and $S_B \neq \emptyset$. Second, we can see that $\sum_{\boldsymbol{s}' \in S_B^*} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0^*, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N) > 0$, as long as $S_B^* \neq \emptyset$. Since $S_B^* = \emptyset$ cannot be the solution to the optimization problem (30) (because $S_B^* = \emptyset$ violates the constraints), we know that $\zeta(\varepsilon) > 0$.

A.2. Proof of the Claim

We prove that for any self-generating set $(\mathcal{W}^{\boldsymbol{\theta}})_{\boldsymbol{\theta} \in \Theta^N}$, the maximum payoff in $(\mathcal{W}^{\boldsymbol{\theta}})_{\boldsymbol{\theta} \in \Theta^N}$, namely, $\max_{\boldsymbol{\theta} \in \Theta^N} \max_{\boldsymbol{v} \in \mathcal{W}^{\boldsymbol{\theta}}} \max_{i \in \mathcal{N}} v_i$, is bounded away from the social optimum $b - c$, regardless of the discount factor. In this way, we can prove that any equilibrium payoff is bounded away from the social optimum. In addition, we analytically quantify the efficiency loss, which is independent of the discount factor.

Since the strategies are restricted on the subset of plans A^{as}, in each period, all the users will receive the same stage-game payoff, either $(b - c)$ or 0, regardless of the matching rule and the rating profile. Hence, the expected discounted average payoff for each user is the same. More precisely, at any given history $\boldsymbol{h}^t = (\boldsymbol{\theta}^0, \ldots, \boldsymbol{\theta}^t)$, we have

$$U_i(\boldsymbol{\theta}^t, \pi_0|_{\boldsymbol{h}^t}, \pi|_{\boldsymbol{h}^t} \cdot \mathbf{1}_N) = U_j(\boldsymbol{\theta}^t, \pi_0|_{\boldsymbol{h}^t}, \pi|_{\boldsymbol{h}^t} \cdot \mathbf{1}_N), \ \forall i, j \in \mathcal{N}, \tag{32}$$

for any $(\pi_0, \pi \cdot \mathbf{1}_N) \in \Pi(A^{\mathrm{as}}) \times \Pi^N(A^{\mathrm{as}})$. As a result, when we restrict to the plan set A^{as}, the self-generating set $(\mathcal{W}^{\boldsymbol{\theta}})_{\boldsymbol{\theta} \in \Theta^N}$ satisfies for any $\boldsymbol{\theta}$ and any $\boldsymbol{v} \in \mathcal{W}^{\boldsymbol{\theta}}$.

$$v_i = v_j, \ \forall i, j \in \mathcal{N}. \tag{33}$$

Given any self-generating set $(\mathcal{W}^{\boldsymbol{\theta}})_{\boldsymbol{\theta} \in \Theta^N}$, define the maximum payoff $\bar{v}$ as

$$\bar{v} \triangleq \max_{\boldsymbol{\theta} \in \Theta^N} \max_{\boldsymbol{v} \in \mathcal{W}^{\boldsymbol{\theta}}} \max_{i \in \mathcal{N}} v_i. \tag{34}$$

Now we derive the upper bound of $\bar{v}$ by looking at the decomposability constraints.

To decompose the payoff profile $\bar{v} \cdot \mathbf{1}_N$, we must find a recommended plan $\alpha_0 \in A^{\mathrm{as}}$, a plan profile $\alpha \cdot \mathbf{1}_N$ with $\alpha \in A^{\mathrm{as}}$, and a continuation payoff function $\boldsymbol{\gamma} : \Theta^N \to \cup_{\boldsymbol{\theta}' \in \Theta^N} \mathcal{W}^{\boldsymbol{\theta}'}$

with $\boldsymbol{\gamma}(\boldsymbol{\theta}') \in \mathcal{W}^{\boldsymbol{\theta}'}$, such that for all $i \in \mathcal{N}$ and for all $\alpha_i \in A$,

$$\begin{aligned} \bar{v} \;&=\; (1-\delta)u_i(\boldsymbol{\theta}, \alpha_0, \alpha \cdot \mathbf{1}_N) + \delta \sum_{\boldsymbol{\theta}'} \gamma_i(\boldsymbol{\theta}') q(\boldsymbol{\theta}'|\boldsymbol{\theta}, \alpha_0, \alpha \cdot \mathbf{1}_N) \\ &\geq\; (1-\delta)u_i(\boldsymbol{\theta}, \alpha_0, \alpha_i, \alpha \cdot \mathbf{1}_{N-1}) + \delta \sum_{\boldsymbol{\theta}'} \gamma_i(\boldsymbol{\theta}') q(\boldsymbol{\theta}'|\boldsymbol{\theta}, \alpha_0, \alpha_i, \alpha \cdot \mathbf{1}_{N-1}). \end{aligned} \tag{35}$$

Note that we do not require the users' plan α to be the same as the recommended plan α_0, and that we also do not require the continuation payoff function $\boldsymbol{\gamma}$ to be a simple continuation payoff function.

First, the payoff profile $\bar{v} \cdot \mathbf{1}_N$ cannot be decomposed by a recommended plan α_0 and the selfish plan $\alpha^{\rm s}$. Otherwise, since $\boldsymbol{\gamma}(\boldsymbol{\theta}') \in \mathcal{W}^{\boldsymbol{\theta}'}$, we have

$$\bar{v} = (1-\delta)\cdot 0 + \delta \sum_{\boldsymbol{\theta}'} \gamma_i(\boldsymbol{\theta}') q(\boldsymbol{\theta}'|\boldsymbol{\theta}, \alpha_0, \alpha^{\rm a} \cdot \mathbf{1}_N) \leq \delta \sum_{\boldsymbol{\theta}'} \bar{v}_i \cdot q(\boldsymbol{\theta}'|\boldsymbol{\theta}, \alpha_0, \alpha^{\rm a} \cdot \mathbf{1}_N) = \delta \cdot \bar{v} < \bar{v},$$

which is a contradiction.

Since we must use a recommended plan α_0 and the altruistic plan $\alpha^{\rm a}$ to decompose $\bar{v} \cdot \mathbf{1}_N$, we can rewrite the decomposability constraint as

$$\begin{aligned} \bar{v} \;&=\; (1-\delta)(b-c) + \delta \sum_{\boldsymbol{\theta}'} \gamma_i(\boldsymbol{\theta}') q(\boldsymbol{\theta}'|\boldsymbol{\theta}, \alpha_0, \alpha^{\rm a} \cdot \mathbf{1}_N) \\ &\geq\; (1-\delta)u_i(\boldsymbol{\theta}, \alpha_0, \alpha_i, \alpha^{\rm a} \cdot \mathbf{1}_{N-1}) + \delta \sum_{\boldsymbol{\theta}'} \gamma_i(\boldsymbol{\theta}') q(\boldsymbol{\theta}'|\boldsymbol{\theta}, \alpha_0, \alpha_i, \alpha^{\rm a} \cdot \mathbf{1}_{N-1}). \end{aligned} \tag{36}$$

Since the continuation payoffs under different rating profiles $\boldsymbol{\theta}, \boldsymbol{\theta}'$ that have the same rating distribution $\boldsymbol{s}(\boldsymbol{\theta}) = \boldsymbol{s}(\boldsymbol{\theta}')$ are the same, namely, $\boldsymbol{\gamma}(\boldsymbol{\theta}) = \boldsymbol{\gamma}(\boldsymbol{\theta}')$, the continuation payoff depends only on the rating distribution. For notational simplicity, with some abuse of notation, we write $\boldsymbol{\gamma}(\boldsymbol{s})$ as the continuation payoff when the rating distribution is $\boldsymbol{s}$, write $q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha_i, \alpha^{\rm a} \cdot \mathbf{1}_{N-1})$ as the probability that the next state has a rating distribution $\boldsymbol{s}'$, and write $u_i(\boldsymbol{s}, \alpha^{\rm a}, \alpha_i, \alpha^{\rm a} \cdot \mathbf{1}_{N-1})$ as the stage-game payoff when the next state has a rating distribution $\boldsymbol{s}$. Then, the decomposability constraint can be rewritten as

$$\begin{aligned} \bar{v} \;&=\; (1-\delta)(b-c) + \delta \sum_{\boldsymbol{s}'} \gamma_i(\boldsymbol{s}') q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha^{\rm a} \cdot \mathbf{1}_N) \\ &\geq\; (1-\delta)u_i(\boldsymbol{s}, \alpha_0, \alpha_i, \alpha^{\rm a} \cdot \mathbf{1}_{N-1}) + \delta \sum_{\boldsymbol{s}'} \gamma_i(\boldsymbol{s}') q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha_i, \alpha^{\rm a} \cdot \mathbf{1}_{N-1}). \end{aligned} \tag{37}$$

Now we focus on a subclass of continuation payoff functions and derive the maximum payoff $\bar{v}$ achievable under this subclass of continuation payoff functions. Later, we will prove that we cannot increase $\bar{v}$ by choosing other continuation payoff functions. Specifically, we focus on a subclass of continuation payoff functions that satisfy

$$\gamma_i(\boldsymbol{s}) \;=\; x_A,\ \forall i \in \mathcal{N},\ \forall \boldsymbol{s} \in S_A \subset S, \tag{38}$$

$$\gamma_i(\boldsymbol{s}) \;=\; x_B,\ \forall i \in \mathcal{N},\ \forall \boldsymbol{s} \in S_B \subset S, \tag{39}$$

where S_A and S_B are subsets of the set of rating distributions S that have no intersection, namely, $S_A \cap S_B = \emptyset$. In other words, we assign the two continuation payoff values to two subsets of rating distributions, respectively. Without loss of generality, we assume $x_A \geq x_B$.

Now we derive the incentive compatibility constraints. There are three plans to deviate to: the plan α^0 in which the user does not serve users with rating 0, the plan α^1 in which the user does not serve users with rating 1, and the plan α^{01} in which the user does not serve anyone. The corresponding incentive compatibility constraints for a user i with rating $\theta_i = 1$ are

$$\left[\sum_{\boldsymbol{s}' \in S_A} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N) - q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha_i = \alpha^0, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N)\right](x_A - x_B) \geq \frac{1-\delta}{\delta}\frac{s_0}{N-1}c,$$
$$\left[\sum_{\boldsymbol{s}' \in S_A} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N) - q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha_i = \alpha^1, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N)\right](x_A - x_B) \geq \frac{1-\delta}{\delta}\frac{s_1-1}{N-1}c,$$
$$\left[\sum_{\boldsymbol{s}' \in S_A} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N) - q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha_i = \alpha^{01}, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N)\right](x_A - x_B) \geq \frac{1-\delta}{\delta}c. \tag{40}$$

Similarly, the corresponding incentive compatibility constraints for a user j with rating $\theta_j = 0$ are

$$\left[\sum_{\boldsymbol{s}' \in S_A} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N) - q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha_j = \alpha^0, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N)\right](x_A - x_B) \geq \frac{1-\delta}{\delta}\frac{s_0-1}{N-1}c,$$
$$\left[\sum_{\boldsymbol{s}' \in S_A} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N) - q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha_j = \alpha^1, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N)\right](x_A - x_B) \geq \frac{1-\delta}{\delta}\frac{s_1}{N-1}c,$$
$$\left[\sum_{\boldsymbol{s}' \in S_A} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N) - q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha_j = \alpha^{01}, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N)\right](x_A - x_B) \geq \frac{1-\delta}{\delta}c. \tag{41}$$

We can summarize these incentive compatibility constraints as

$$x_A - x_B \geq \frac{1-\delta}{\delta}c \cdot \rho(\boldsymbol{\theta}, \alpha_0, S_A), \tag{42}$$

where

$$\rho(\boldsymbol{\theta}, \alpha_0, S_B) \triangleq \max_{i \in \mathcal{N}} \max \left\{ \frac{\frac{s_{\theta_i}-1}{N-1}}{\sum_{\boldsymbol{s}' \in S \setminus S_B} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N) - q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha_i = \alpha^0, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N)}, \right. \tag{43}$$
$$\frac{\frac{s_{1-\theta_i}}{N-1}}{\sum_{\boldsymbol{s}' \in S \setminus S_B} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N) - q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha_i = \alpha^1, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N)}, \tag{44}$$
$$\left. \frac{1}{\sum_{\boldsymbol{s}' \in S \setminus S_B} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N) - q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha_i = \alpha^{01}, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N)} \right\}. \tag{45}$$

Since the maximum payoff $\bar{v}$ satisfies

$$\bar{v} = (1-\delta)(b-c) + \delta\left(x_A \sum_{\boldsymbol{s}' \in S \setminus S_B} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N) + x_B \sum_{\boldsymbol{s}' \in S_B} q(\boldsymbol{s}'|\boldsymbol{\theta}, \alpha_0, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N)\right), \tag{46}$$

to maximize $\bar{v}$, we choose $x_B = x_A - \frac{1-\delta}{\delta} c \cdot \rho(\boldsymbol{\theta}, \alpha_0, S_B)$. Since $x_A \geq \bar{v}$, we have

$$\bar{v} = (1-\delta)(b-c) + \delta \left(x_A - \frac{1-\delta}{\delta} c \cdot \rho(\boldsymbol{\theta}, \alpha_0, S_B) \sum_{\boldsymbol{s}' \in S_B} q(\boldsymbol{s}' | \boldsymbol{\theta}, \alpha_0, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N) \right) \quad (47)$$

$$\leq (1-\delta)(b-c) + \delta \left(\bar{v} - \frac{1-\delta}{\delta} c \cdot \rho(\boldsymbol{\theta}, \alpha_0, S_B) \sum_{\boldsymbol{s}' \in S_B} q(\boldsymbol{s}' | \boldsymbol{\theta}, \alpha_0, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N) \right), \quad (48)$$

which leads to

$$\bar{v} \leq b - c - c \cdot \rho(\boldsymbol{\theta}, \alpha_0, S_B) \sum_{\boldsymbol{s}' \in S_B} q(\boldsymbol{s}' | \boldsymbol{\theta}, \alpha_0, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N). \quad (49)$$

Hence, the maximum payoff $\bar{v}$ satisfies

$$\bar{v} \leq b - c - c \cdot \min_{S_B \subset S} \left\{ \rho(\boldsymbol{\theta}, \alpha_0, S_B) \sum_{\boldsymbol{s}' \in S_B} q(\boldsymbol{s}' | \boldsymbol{\theta}, \alpha_0, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N) \right\}, \quad (50)$$

where S_B satisfies for all $i \in \mathcal{N}$,

$$\begin{aligned}
\sum_{\boldsymbol{s}' \in S \setminus S_B} q(\boldsymbol{s}' | \boldsymbol{\theta}, \alpha_0, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N) &> \sum_{\boldsymbol{s}' \in S \setminus S_B} q(\boldsymbol{s}' | \boldsymbol{\theta}, \alpha_0, \alpha_i = \alpha^0, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N), \\
\sum_{\boldsymbol{s}' \in S \setminus S_B} q(\boldsymbol{s}' | \boldsymbol{\theta}, \alpha_0, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N) &> \sum_{\boldsymbol{s}' \in S \setminus S_B} q(\boldsymbol{s}' | \boldsymbol{\theta}, \alpha_0, \alpha_i = \alpha^1, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N), \\
\sum_{\boldsymbol{s}' \in S \setminus S_B} q(\boldsymbol{s}' | \boldsymbol{\theta}, \alpha_0, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N) &> \sum_{\boldsymbol{s}' \in S \setminus S_B} q(\boldsymbol{s}' | \boldsymbol{\theta}, \alpha_0, \alpha_i = \alpha^{01}, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N). \quad (51)
\end{aligned}$$

Following the same logic as in the proof of Proposition 6 in Dellarocas [2005], we can prove that we cannot achieve a higher maximum payoff by other continuation payoff functions.

B. PROOF OF THEOREM 5.2

B.1. Outline of the Proof

We derive the conditions under which the set $(\mathcal{W}^{\boldsymbol{\theta}})_{\boldsymbol{\theta} \in \Theta^N}$ is a self-generating set. Specifically, we derive the conditions under which any payoff profile $\boldsymbol{v} \in \mathcal{W}^{\boldsymbol{\theta}}$ is decomposable on $(\mathcal{W}^{\boldsymbol{\theta}'})_{\boldsymbol{\theta}' \in \Theta^N}$ given $\boldsymbol{\theta}$, for all $\boldsymbol{\theta} \in \Theta^N$.

B.2. When Users Have Different Ratings

B.2.1. Preliminaries. We first focus on the states $\boldsymbol{\theta}$ with $1 \leq s_1(\boldsymbol{\theta}) \leq N-1$ and derive the conditions under which any payoff profile $\boldsymbol{v} \in \mathcal{W}^{\boldsymbol{\theta}}$ can be decomposed by $(\alpha_0 = \alpha^{\mathrm{a}}, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N)$ or $(\alpha_0 = \alpha^{\mathrm{f}}, \alpha^{\mathrm{f}} \cdot \mathbf{1}_N)$. First, $\boldsymbol{v}$ could be decomposed by $(\alpha^{\mathrm{a}}, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N)$, if there exists a continuation payoff function $\boldsymbol{\gamma} : \Theta^N \to \cup_{\boldsymbol{\theta}' \in \Theta^N} \mathcal{W}^{\boldsymbol{\theta}'}$ with $\boldsymbol{\gamma}(\boldsymbol{\theta}') \in \mathcal{W}^{\boldsymbol{\theta}'}$, such that for all $i \in \mathcal{N}$ and for all $\alpha_i \in A$,

$$\begin{aligned}
v_i &= (1-\delta) u_i(\boldsymbol{\theta}, \alpha^{\mathrm{a}}, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N) + \delta \sum_{\boldsymbol{\theta}'} \gamma_i(\boldsymbol{\theta}') q(\boldsymbol{\theta}' | \boldsymbol{\theta}, \alpha^{\mathrm{a}}, \alpha^{\mathrm{a}} \cdot \mathbf{1}_N) \quad (52) \\
&\geq (1-\delta) u_i(\boldsymbol{\theta}, \alpha^{\mathrm{a}}, \alpha_i, \alpha^{\mathrm{a}} \cdot \mathbf{1}_{N-1}) + \delta \sum_{\boldsymbol{\theta}'} \gamma_i(\boldsymbol{\theta}') q(\boldsymbol{\theta}' | \boldsymbol{\theta}, \alpha^{\mathrm{a}}, \alpha_i, \alpha^{\mathrm{a}} \cdot \mathbf{1}_{N-1}).
\end{aligned}$$

Since we focus on simple continuation payoff functions, all the users with the same future rating will have the same continuation payoff regardless of the recommended plan α_0, the plan profile $(\alpha_i, \alpha \cdot \mathbf{1}_{N-1})$, and the future state $\boldsymbol{\theta}'$. Hence, we write the continuation payoffs for the users with future rating 1 and 0 as γ^1 and γ^0, respectively. Consequently, the preceding conditions on decomposability can be simplified to

$$\begin{aligned} v_i &= (1-\delta)\cdot u_i(\boldsymbol{\theta}, \alpha^{\mathrm{a}}, \alpha^{\mathrm{a}}\cdot \mathbf{1}_N) \\ &\quad + \delta\left(\gamma^1 \sum_{\boldsymbol{\theta}':\theta_i'=1} q(\boldsymbol{\theta}'|\boldsymbol{\theta}, \alpha^{\mathrm{a}}, \alpha^{\mathrm{a}}\cdot\mathbf{1}_N) + \gamma^0 \sum_{\boldsymbol{\theta}':\theta_i'=0} q(\boldsymbol{\theta}'|\boldsymbol{\theta}, \alpha^{\mathrm{a}}, \alpha^{\mathrm{a}}\cdot\mathbf{1}_N)\right) \\ &\geq (1-\delta)\cdot u_i(\boldsymbol{\theta}, \alpha^{\mathrm{a}}, \alpha_i, \alpha^{\mathrm{a}}\cdot\mathbf{1}_{N-1}) \\ &\quad + \delta\left(\gamma^1 \sum_{\boldsymbol{\theta}':\theta_i'=1} q(\boldsymbol{\theta}'|\boldsymbol{\theta}, \alpha^{\mathrm{a}}, \alpha_i, \alpha^{\mathrm{a}}\cdot\mathbf{1}_{N-1}) + \gamma^0 \sum_{\boldsymbol{\theta}':\theta_i'=0} q(\boldsymbol{\theta}'|\boldsymbol{\theta}, \alpha^{\mathrm{a}}, \alpha_i, \alpha^{\mathrm{a}}\cdot\mathbf{1}_{N-1})\right). \end{aligned} \tag{53}$$

First, consider the case when user i has rating 1 (i.e., $\theta_i = 1$). The stage-game payoff is $u_i(\boldsymbol{\theta}, \alpha^{\mathrm{a}}, \alpha^{\mathrm{a}}\cdot\mathbf{1}_N) = b - c$. The term $\sum_{\boldsymbol{\theta}':\theta_i'=1} q(\boldsymbol{\theta}'|\boldsymbol{\theta}, \alpha^{\mathrm{a}}, \alpha^{\mathrm{a}}\cdot\mathbf{1}_N)$ is the probability that user i has rating 1 in the next state. Since user i's rating update is independent of the other users' rating update, we can calculate this probability as

$$\sum_{\boldsymbol{\theta}':\theta_i'=1} q(\boldsymbol{\theta}'|\boldsymbol{\theta}, \alpha^{\mathrm{a}}, \alpha^{\mathrm{a}}\cdot\mathbf{1}_N) = [(1-\varepsilon)\beta_1^+ + \varepsilon(1-\beta_1^-)] \sum_{m\in M:\theta_{m(i)}=1} \mu(m) \tag{54}$$

$$+ [(1-\varepsilon)\beta_1^+ + \varepsilon(1-\beta_1^-)] \sum_{m\in M:\theta_{m(i)}=0} \mu(m) \tag{55}$$

$$= (1-\varepsilon)\beta_1^+ + \varepsilon(1-\beta_1^-) = x_1^+. \tag{56}$$

Similarly, we can calculate $\sum_{\boldsymbol{\theta}':\theta_i'=0} q(\boldsymbol{\theta}'|\boldsymbol{\theta}, \alpha^{\mathrm{a}}, \alpha^{\mathrm{a}}\cdot\mathbf{1}_N)$, the probability that user i has rating 0 in the next state, as

$$\sum_{\boldsymbol{\theta}':\theta_i'=0} q(\boldsymbol{\theta}'|\boldsymbol{\theta}, \alpha^{\mathrm{a}}, \alpha^{\mathrm{a}}\cdot\mathbf{1}_N) = [(1-\varepsilon)(1-\beta_1^+) + \varepsilon\beta_1^-] \sum_{m\in M:\theta_{m(i)}=1} \mu(m) \tag{57}$$

$$+ [(1-\varepsilon)(1-\beta_1^+) + \varepsilon\beta_1^-] \sum_{m\in M:\theta_{m(i)}=0} \mu(m) \tag{58}$$

$$= (1-\varepsilon)(1-\beta_1^+) + \varepsilon\beta_1^- = 1 - x_1^+. \tag{59}$$

Now we discuss what happens if user i deviates. Since the recommended plan α^{a} is to exert high effort for all the users, user i can deviate to the other three plans, namely, "exert high effort for rating-1 users only," "exert high effort for rating-0 users only," "exert low effort for all the users." We can calculate the corresponding stage-game payoff and state transition probabilities under each deviation.

—*Exert high effort for rating-1 users only* $(\alpha_i(1, \theta_i) = 1, \alpha_i(0, \theta_i) = 0)$:

$$u_i(\boldsymbol{\theta}, \alpha^{\mathrm{a}}, \alpha_i, \alpha^{\mathrm{a}}\cdot\mathbf{1}_{N-1}) = b - c\cdot \sum_{m\in M:\theta_{m(i)}=1} \mu(m) = b - c\cdot \frac{s_1(\boldsymbol{\theta}) - 1}{N-1} \tag{60}$$

$$\sum_{\boldsymbol{\theta}':\theta_i'=1} q(\boldsymbol{\theta}'|\boldsymbol{\theta},\alpha^{\mathrm{a}},\alpha_i,\alpha^{\mathrm{a}}\cdot\mathbf{1}_{N-1}) \tag{61}$$
$$= [(1-\varepsilon)\beta_1^+ + \varepsilon(1-\beta_1^-)] \sum_{m\in M:\theta_{m(i)}=1} \mu(m) + [(1-\varepsilon)(1-\beta_1^-)+\varepsilon\beta_1^+] \sum_{m\in M:\theta_{m(i)}=0} \mu(m)$$
$$= [(1-\varepsilon)\beta_1^+ + \varepsilon(1-\beta_1^-)] \frac{s_1(\boldsymbol{\theta})-1}{N-1} + [(1-\varepsilon)(1-\beta_1^-)+\varepsilon\beta_1^+] \frac{s_0(\boldsymbol{\theta})}{N-1}.$$

$$\sum_{\boldsymbol{\theta}':\theta_i'=0} q(\boldsymbol{\theta}'|\boldsymbol{\theta},\alpha^{\mathrm{a}},\alpha_i,\alpha^{\mathrm{a}}\cdot\mathbf{1}_{N-1}) \tag{62}$$
$$= [(1-\varepsilon)(1-\beta_1^+) + \varepsilon\beta_1^-] \sum_{m\in M:\theta_{m(i)}=1} \mu(m) + [(1-\varepsilon)\beta_1^- + \varepsilon(1-\beta_1^+)] \sum_{m\in M:\theta_{m(i)}=0} \mu(m)$$
$$= [(1-\varepsilon)(1-\beta_1^+) + \varepsilon\beta_1^-] \frac{s_1(\boldsymbol{\theta})-1}{N-1} + [(1-\varepsilon)\beta_1^- + \varepsilon(1-\beta_1^+)] \frac{s_0(\boldsymbol{\theta})}{N-1}.$$

—*Exert high effort for rating-0 users only* ($\alpha_i(1,\theta_i)=0, \alpha_i(0,\theta_i)=1$):

$$u_i(\boldsymbol{\theta},\alpha^{\mathrm{a}},\alpha_i,\alpha^{\mathrm{a}}\cdot\mathbf{1}_{N-1}) \;=\; b - c\cdot \sum_{m\in M:\theta_{m(i)}=0} \mu(m) = b - c\cdot\frac{s_0(\boldsymbol{\theta})}{N-1} \tag{63}$$

$$\sum_{\boldsymbol{\theta}':\theta_i'=1} q(\boldsymbol{\theta}'|\boldsymbol{\theta},\alpha^{\mathrm{a}},\alpha_i,\alpha^{\mathrm{a}}\cdot\mathbf{1}_{N-1}) \tag{64}$$
$$= [(1-\varepsilon)(1-\beta_1^-) + \varepsilon\beta_1^+] \sum_{m\in M:\theta_{m(i)}=1} \mu(m) + [(1-\varepsilon)\beta_1^+ + \varepsilon(1-\beta_1^-)] \sum_{m\in M:\theta_{m(i)}=0} \mu(m)$$
$$= [(1-\varepsilon)(1-\beta_1^-) + \varepsilon\beta_1^+] \frac{s_1(\boldsymbol{\theta})-1}{N-1} + [(1-\varepsilon)\beta_1^+ + \varepsilon(1-\beta_1^-)] \frac{s_0(\boldsymbol{\theta})}{N-1}.$$

$$\sum_{\boldsymbol{\theta}':\theta_i'=0} q(\boldsymbol{\theta}'|\boldsymbol{\theta},\alpha^{\mathrm{a}},\alpha_i,\alpha^{\mathrm{a}}\cdot\mathbf{1}_{N-1}) \tag{65}$$
$$= [(1-\varepsilon)\beta_1^- + \varepsilon(1-\beta_1^+)] \sum_{m\in M:\theta_{m(i)}=1} \mu(m) + [(1-\varepsilon)(1-\beta_1^+) + \varepsilon\beta_1^-] \sum_{m\in M:\theta_{m(i)}=0} \mu(m)$$
$$= [(1-\varepsilon)\beta_1^- + \varepsilon(1-\beta_1^+)] \frac{s_1(\boldsymbol{\theta})-1}{N-1} + [(1-\varepsilon)(1-\beta_1^+) + \varepsilon\beta_1^-] \frac{s_0(\boldsymbol{\theta})}{N-1}.$$

—*Exert low effort for all the users* ($\alpha_i(1,\theta_i)=0, \alpha_i(0,\theta_i)=0$):

$$u_i(\boldsymbol{\theta},\alpha^{\mathrm{a}},\alpha_i,\alpha^{\mathrm{a}}\cdot\mathbf{1}_{N-1}) \;=\; b \tag{66}$$

$$\sum_{\boldsymbol{\theta}':\theta_i'=1} q(\boldsymbol{\theta}'|\boldsymbol{\theta},\alpha^{\mathrm{a}},\alpha_i,\alpha^{\mathrm{a}}\cdot\mathbf{1}_{N-1}) \tag{67}$$
$$= [(1-\varepsilon)(1-\beta_1^-) + \varepsilon\beta_1^+] \sum_{m\in M:\theta_{m(i)}=1} \mu(m) + [(1-\varepsilon)(1-\beta_1^-) + \varepsilon\beta_1^+] \sum_{m\in M:\theta_{m(i)}=0} \mu(m)$$
$$= (1-\varepsilon)(1-\beta_1^-) + \varepsilon\beta_1^+.$$

$$\sum_{\boldsymbol{\theta}':\theta_i'=0} q(\boldsymbol{\theta}'|\boldsymbol{\theta}, \alpha^{\mathrm{a}}, \alpha_i, \alpha^{\mathrm{a}} \cdot \mathbf{1}_{N-1}) \tag{68}$$
$$= [(1-\varepsilon)\beta_1^- + \varepsilon(1-\beta_1^+)] \sum_{m\in M:\theta_{m(i)}=1} \mu(m) + [(1-\varepsilon)\beta_1^- + \varepsilon(1-\beta_1^+)] \sum_{m\in M:\theta_{m(i)}=0} \mu(m)$$
$$= (1-\varepsilon)\beta_1^- + \varepsilon(1-\beta_1^+).$$

Plugging these expressions into Eq. (53), we can simplify the incentive compatibility constraints (i.e., the inequality constraints) to

$$(1-2\varepsilon)\left[\beta_1^+ - (1-\beta_1^-)\right](\gamma^1 - \gamma^0) \geq \frac{1-\delta}{\delta} \cdot c, \tag{69}$$

under all three deviating plans.

Hence, if user i has rating 1, the decomposability constraints (53) reduce to

$$v^1 = (1-\delta)\cdot(b-c) + \delta \cdot \left[x_1^+\gamma^1 + (1-x_1^+)\gamma^0\right], \tag{70}$$

where v^1 is the payoff of the users with rating 1, and

$$(1-2\varepsilon)\left[\beta_1^+ - (1-\beta_1^-)\right](\gamma^1 - \gamma^0) \geq \frac{1-\delta}{\delta} \cdot c. \tag{71}$$

Similarly, if user i has rating 0, we can reduce the decomposability constraints (53) to

$$v^0 = (1-\delta)\cdot(b-c) + \delta \cdot \left[x_0^+\gamma^1 + (1-x_0^+)\gamma^0\right], \tag{72}$$

and

$$(1-2\varepsilon)\left[\beta_0^+ - (1-\beta_0^-)\right](\gamma^1 - \gamma^0) \geq \frac{1-\delta}{\delta} \cdot c. \tag{73}$$

For these incentive compatibility constraints (the preceding two inequalities) to hold, we need to have $\beta_1^+ - (1-\beta_1^-) > 0$ and $\beta_0^+ - (1-\beta_0^-) > 0$, which are part of Condition 1 and Condition 2. Now we will derive the rest of the sufficient conditions in Theorem 5.2.

The preceding two equalities determine the continuation payoff γ^1 and γ^0, as follows:

$$\begin{cases} \gamma^1 = \frac{1}{\delta} \cdot \frac{(1-x_0^+)v^1 - (1-x_1^+)v^0}{x_1^+ - x_0^+} - \frac{1-\delta}{\delta} \cdot (b-c) \\ \gamma^0 = \frac{1}{\delta} \cdot \frac{x_1^+ v^0 - x_0^+ v^1}{x_1^+ - x_0^+} - \frac{1-\delta}{\delta} \cdot (b-c) \end{cases}. \tag{74}$$

Now we consider the decomposability constraints if we want to decompose a payoff profile $\boldsymbol{v} \in \mathcal{W}^{\boldsymbol{\theta}}$ using the fair plan α^{f}. Since we focus on decomposition by simple continuation payoff functions, we write the decomposition constraints as

$$\begin{aligned} v_i \;=\; & (1-\delta)\cdot u_i(\boldsymbol{\theta}, \alpha^{\mathrm{f}}, \alpha^{\mathrm{f}} \cdot \mathbf{1}_N) \\ & + \delta \left(\gamma^1 \sum_{\boldsymbol{\theta}':\theta_i'=1} q(\boldsymbol{\theta}'|\boldsymbol{\theta}, \alpha^{\mathrm{f}}, \alpha^{\mathrm{f}} \cdot \mathbf{1}_N) + \gamma^0 \sum_{\boldsymbol{\theta}':\theta_i'=0} q(\boldsymbol{\theta}'|\boldsymbol{\theta}, \alpha^{\mathrm{f}}, \alpha^{\mathrm{f}} \cdot \mathbf{1}_N) \right) \\ \geq\; & (1-\delta)\cdot u_i(\boldsymbol{\theta}, \alpha^{\mathrm{f}}, \alpha_i, \alpha^{\mathrm{f}} \cdot \mathbf{1}_{N-1}) \\ & + \delta \left(\gamma^1 \sum_{\boldsymbol{\theta}':\theta_i'=1} q(\boldsymbol{\theta}'|\boldsymbol{\theta}, \alpha^{\mathrm{f}}, \alpha_i, \alpha^{\mathrm{f}} \cdot \mathbf{1}_{N-1}) + \gamma^0 \sum_{\boldsymbol{\theta}':\theta_i'=0} q(\boldsymbol{\theta}'|\boldsymbol{\theta}, \alpha^{\mathrm{f}}, \alpha_i, \alpha^{\mathrm{f}} \cdot \mathbf{1}_{N-1}) \right). \end{aligned} \tag{75}$$

Due to space limitation, we omit the details and directly give the simplification of the preceding decomposability constraints as follows. First, the incentive compatibility constraints (i.e., the inequality constraints) are simplified to

$$(1-2\varepsilon)\left[\beta_1^+ - (1-\beta_1^-)\right](\gamma^1-\gamma^0) \geq \frac{1-\delta}{\delta}\cdot c, \tag{76}$$

and

$$(1-2\varepsilon)\left[\beta_0^+ - (1-\beta_0^-)\right](\gamma^1-\gamma^0) \geq \frac{1-\delta}{\delta}\cdot c, \tag{77}$$

under all three deviating plans. Note that the preceding incentive compatibility constraints are the same as the ones when we want to decompose the payoffs using the altruistic plan α^{a}.

Then, the equality constraints in Eq. (75) can be simplified as follows. For the users with rating 1, we have

$$v^1 = (1-\delta)\cdot\left(b - \frac{s_1(\boldsymbol{\theta})-1}{N-1}c\right) + \delta\cdot\left[x^+_{s_1(\boldsymbol{\theta})}\cdot\gamma^1 + (1-x^+_{s_1(\boldsymbol{\theta})})\cdot\gamma^0\right], \tag{78}$$

where

$$x_{s_1(\boldsymbol{\theta})} \triangleq \left[(1-\varepsilon)\frac{s_1(\boldsymbol{\theta})-1}{N-1} + \frac{s_0(\boldsymbol{\theta})}{N-1}\right]\beta_1^+ + \left(\varepsilon\frac{s_1(\boldsymbol{\theta})-1}{N-1}\right)(1-\beta_1^-). \tag{79}$$

For the users with rating 0, we have

$$v^0 = (1-\delta)\cdot\left(\frac{s_0(\boldsymbol{\theta})-1}{N-1}b - c\right) + \delta\cdot\left[x_0^+\gamma^1 + (1-x_0^+)\gamma^0\right]. \tag{80}$$

These two equalities determine the continuation payoff γ^1 and γ^0, as follows:

$$\begin{cases} \gamma^1 = \frac{1}{\delta}\cdot\frac{(1-x_0^+)v^1-(1-x^+_{s_1(\boldsymbol{\theta})})v^0}{x^+_{s_1(\boldsymbol{\theta})}-x_0^+} - \frac{1-\delta}{\delta}\cdot\frac{\left(b-\frac{s_1(\boldsymbol{\theta})-1}{N-1}c\right)(1-x_0^+)-\left(\frac{s_0(\boldsymbol{\theta})-1}{N-1}b-c\right)(1-x^+_{s_1(\boldsymbol{\theta})})}{x^+_{s_1(\boldsymbol{\theta})}-x_0^+} \\ \gamma^0 = \frac{1}{\delta}\cdot\frac{x^+_{s_1(\boldsymbol{\theta})}v^0-x_0^+v^1}{x^+_{s_1(\boldsymbol{\theta})}-x_0^+} - \frac{1-\delta}{\delta}\cdot\frac{\left(b-\frac{s_1(\boldsymbol{\theta})-1}{N-1}c\right)x_0^+-\left(\frac{s_0(\boldsymbol{\theta})-1}{N-1}b-c\right)x^+_{s_1(\boldsymbol{\theta})}}{x^+_{s_1(\boldsymbol{\theta})}-x_0^+} \end{cases}. \tag{81}$$

B.2.2. Sufficient Conditions. Now we derive the sufficient conditions under which any payoff profile $\boldsymbol{v} \in \mathcal{W}^{\boldsymbol{\theta}}$ can be decomposed by $(\alpha_0 = \alpha^{\mathrm{a}}, \alpha^{\mathrm{a}}\cdot\mathbf{1}_N)$ or $(\alpha_0 = \alpha^{\mathrm{f}}, \alpha^{\mathrm{f}}\cdot\mathbf{1}_N)$. Specifically, we will derive the conditions such that for any payoff profile $\boldsymbol{v} \in \mathcal{W}^{\boldsymbol{\theta}}$, at least one of the two decomposability constraints (53) and (75) is satisfied. From the preliminaries, we know that the incentive compatibility constraints in (53) and (75) can be simplified into the same constraints:

$$(1-2\varepsilon)\left[\beta_1^+ - (1-\beta_1^-)\right](\gamma^1-\gamma^0) \geq \frac{1-\delta}{\delta}\cdot c, \tag{82}$$

and

$$(1-2\varepsilon)\left[\beta_0^+ - (1-\beta_0^-)\right](\gamma^1-\gamma^0) \geq \frac{1-\delta}{\delta}\cdot c. \tag{83}$$

These constraints impose the constraint on the discount factor, namely,

$$\delta \geq \max_{\theta\in\Theta}\frac{c}{c+(1-2\varepsilon)\left[\beta_\theta^+ - (1-\beta_\theta^-)\right](\gamma^1-\gamma^0)}. \tag{84}$$

Since γ_1 and γ_0 should satisfy $\gamma^1 - \gamma^0 \geq \epsilon_0 - \epsilon_1$, these constraints can be rewritten as

$$\delta \geq \max_{\theta \in \Theta} \frac{c}{c + (1-2\varepsilon)\left[\beta_\theta^+ - (1-\beta_\theta^-)\right](\epsilon_0 - \epsilon_1)}, \tag{85}$$

where is part of Condition 3 in Theorem 5.2.

In addition, the continuation payoffs γ_1 and γ_0 should satisfy the constraints of the self-generating set, namely,

$$\gamma^1 - \gamma^0 \ \geq \ \epsilon_0 - \epsilon_1, \tag{86}$$

$$\gamma^1 + \frac{c}{(N-1)b} \cdot \gamma^0 \ \leq \ z_2 \triangleq \left(1 + \frac{c}{(N-1)b}\right)(b-c) - \frac{c}{(N-1)b}\epsilon_0 - \epsilon_1, \tag{87}$$

$$\gamma^1 - \frac{b}{\frac{N-2}{N-1}b - c} \cdot \gamma^0 \ \leq \ z_3 \triangleq - \frac{\frac{b}{\frac{N-2}{N-1}b-c} - 1}{1 + \frac{c}{(N-1)b}} \cdot z_2. \tag{88}$$

We can plug the expressions of the continuation payoffs γ_1 and γ_0 in (74) and (81) into the preceding constraints. Specifically, if a payoff profile $\boldsymbol{v}$ is decomposed by the altruistic plan, the following constraints should be satisfied for the continuation payoff profile to be in the self-generating set. (For notational simplicity, we define $\kappa_1 \triangleq \frac{b}{\frac{N-2}{N-1}b-c} - 1$ and $\kappa_2 \triangleq 1 + \frac{c}{(N-1)b}$.)

$$\frac{1}{\delta} \cdot \frac{v^1 - v^0}{x_1^+ - x_0^+} \geq \epsilon_0 - \epsilon_1, \tag{α^{a}-1}$$

$$\frac{1}{\delta} \cdot \left\{ \frac{(1-\kappa_2 x_0^+)v^1 - (1-\kappa_2 x_1^+)v^0}{x_1^+ - x_0^+} - \kappa_2 \cdot (b-c) \right\} \leq z_2 - \kappa_2 \cdot (b-c), \tag{α^{a}-2}$$

$$\frac{1}{\delta} \cdot \left\{ \frac{(1+\kappa_1 x_0^+)v^1 - (1+\kappa_1 x_1^+)v^0}{x_1^+ - x_0^+} + \kappa_1 \cdot (b-c) \right\} \leq z_3 + \kappa_1 \cdot (b-c). \tag{α^{a}-3}$$

The constraint (α^{a}-1) is satisfied for all v^1 and v^0 as long as $x_1^+ > x_0^+$, because $v^1 - v^0 > \epsilon_0 - \epsilon_1$, $|x_1^+ > x_0^+| < 1$, and $\delta < 1$.

Since both the left-hand side (LHS) and the right-hand side (RHS) of (α^{a}-2) are smaller than 0, we have

$$(\alpha^{\mathrm{a}}\text{-2}) \Leftrightarrow \delta \leq \frac{\frac{(1-\kappa_2 x_0^+)v^1 - (1-\kappa_2 x_1^+)v^0}{x_1^+ - x_0^+} - \kappa_2 \cdot (b-c)}{z_2 - \kappa_2 \cdot (b-c)}. \tag{89}$$

The RHS of (α^{a}-3) is larger than 0. Hence, we have

$$(\alpha^{\mathrm{a}}\text{-3}) \Leftrightarrow \delta \geq \frac{\frac{(1+\kappa_1 x_0^+)v^1 - (1+\kappa_1 x_1^+)v^0}{x_1^+ - x_0^+} + \kappa_1 \cdot (b-c)}{z_3 + \kappa_1 \cdot (b-c)}. \tag{90}$$

If a payoff profile $\boldsymbol{v}$ is decomposed by the fair plan, the following constraints should be satisfied for the continuation payoff profile to be in the self-generating set:

$$\frac{1}{\delta}\cdot\left\{\frac{v^1-v^0}{x^+_{s_1(\boldsymbol{\theta})}-x^+_0}-\frac{\frac{s_1(\boldsymbol{\theta})}{N-1}b+\frac{s_0(\boldsymbol{\theta})}{N-1}c}{x^+_{s_1(\boldsymbol{\theta})}-x^+_0}\right\}\geq\epsilon_0-\epsilon_1-\frac{\frac{s_1(\boldsymbol{\theta})}{N-1}b+\frac{s_0(\boldsymbol{\theta})}{N-1}c}{x^+_{s_1(\boldsymbol{\theta})}-x^+_0},\qquad(\alpha^{\mathrm{f}}\text{-}1)$$

$$\frac{1}{\delta}\cdot\left\{\frac{(1-\kappa_2x_0^+)v^1-(1-\kappa_2x^+_{s_1(\boldsymbol{\theta})})v^0}{x^+_{s_1(\boldsymbol{\theta})}-x^+_0}-\frac{(1-\kappa_2x_0^+)\left(b-\frac{s_1(\boldsymbol{\theta})-1}{N-1}c\right)-(1-\kappa_2x^+_{s_1(\boldsymbol{\theta})})\left(\frac{s_0(\boldsymbol{\theta})-1}{N-1}b-c\right)}{x^+_{s_1(\boldsymbol{\theta})}-x^+_0}\right\}$$
$$\leq z_2-\frac{(1-\kappa_2x_0^+)\left(b-\frac{s_1(\boldsymbol{\theta})-1}{N-1}c\right)-(1-\kappa_2x^+_{s_1(\boldsymbol{\theta})})\left(\frac{s_0(\boldsymbol{\theta})-1}{N-1}b-c\right)}{x^+_{s_1(\boldsymbol{\theta})}-x^+_0},\qquad(\alpha^{\mathrm{f}}\text{-}2)$$

$$\frac{1}{\delta}\cdot\left\{\frac{(1+\kappa_1x_0^+)v^1-(1+\kappa_1x^+_{s_1(\boldsymbol{\theta})})v^0}{x^+_{s_1(\boldsymbol{\theta})}-x^+_0}-\frac{(1+\kappa_1x_0^+)\left(b-\frac{s_1(\boldsymbol{\theta})-1}{N-1}c\right)-(1+\kappa_1x^+_{s_1(\boldsymbol{\theta})})\left(\frac{s_0(\boldsymbol{\theta})-1}{N-1}b-c\right)}{x^+_{s_1(\boldsymbol{\theta})}-x^+_0}\right\}$$
$$\leq z_3-\frac{(1+\kappa_1x_0^+)\left(b-\frac{s_1(\boldsymbol{\theta})-1}{N-1}c\right)-(1+\kappa_1x^+_{s_1(\boldsymbol{\theta})})\left(\frac{s_0(\boldsymbol{\theta})-1}{N-1}b-c\right)}{x^+_{s_1(\boldsymbol{\theta})}-x^+_0}.\qquad(\alpha^{\mathrm{f}}\text{-}3)$$

Since $\frac{v^1-v^0}{x^+_{s_1(\boldsymbol{\theta})}-x^+_0}>\epsilon_0-\epsilon_1$, the constraint ($\alpha^{\mathrm{f}}$-1) is satisfied for all v^1 and v^0 if $v^1-v^0\geq\frac{s_1(\boldsymbol{\theta})}{N-1}b+\frac{s_0(\boldsymbol{\theta})}{N-1}c$. Hence, the constraint (α^{f}-1) is equivalent to

$$\delta\geq\frac{v^1-v^0-\left(\frac{s_1(\boldsymbol{\theta})}{N-1}b+\frac{s_0(\boldsymbol{\theta})}{N-1}c\right)}{(\epsilon^0-\epsilon^1)(x^+_{s_1(\boldsymbol{\theta})}-x^+_0)-\left(\frac{s_1(\boldsymbol{\theta})}{N-1}b+\frac{s_0(\boldsymbol{\theta})}{N-1}c\right)},\ \text{for }\boldsymbol{\theta}\ s.t.\ \frac{s_1(\boldsymbol{\theta})}{N-1}b+\frac{s_0(\boldsymbol{\theta})}{N-1}c\geq v^1-v^0.\qquad(91)$$

For (α^{f}-2), we want to make the RHS have the same (minus) sign under any state $\boldsymbol{\theta}$, which is true if

$$1-\kappa_2x_0^+>0,\ 1-\kappa_2x^+_{s_1(\boldsymbol{\theta})}<0,\ \frac{1-\kappa_2x^+_{s_1(\boldsymbol{\theta})}}{1-\kappa_2x_0^+}\geq-(\kappa_2-1),\ s_1(\boldsymbol{\theta})=1,\ldots,N-1,\qquad(92)$$

which leads to

$$x^+_{s_1(\boldsymbol{\theta})}>\frac{1}{\kappa_2},\ x_0^+<\frac{1}{\kappa_2},\ x_0^+<\frac{1-x^+_{s_1(\boldsymbol{\theta})}}{1-\kappa_2},\ s_1(\boldsymbol{\theta})=1,\ldots,N-1,\qquad(93)$$
$$\Leftrightarrow\ \frac{N-2}{N-1}x_1^++\frac{1}{N-1}\beta_1^+>\frac{1}{\kappa_2},\ x_0^+<\min\left\{\frac{1}{\kappa_2},\ \frac{1-\beta_1^+}{1-\kappa_2}\right\}.\qquad(94)$$

Since the RHS of (α^{f}-2) is smaller than 0, we have

$$(\alpha^{\mathrm{f}}\text{-}2)\Leftrightarrow\delta\leq\frac{\frac{(1-\kappa_2x_0^+)v^1-(1-\kappa_2x^+_{s_1(\boldsymbol{\theta})})v^0}{x^+_{s_1(\boldsymbol{\theta})}-x^+_0}-\frac{(1-\kappa_2x_0^+)\left(b-\frac{s_1(\boldsymbol{\theta})-1}{N-1}c\right)-(1-\kappa_2x^+_{s_1(\boldsymbol{\theta})})\left(\frac{s_0(\boldsymbol{\theta})-1}{N-1}b-c\right)}{x^+_{s_1(\boldsymbol{\theta})}-x^+_0}}{z_2-\frac{(1-\kappa_2x_0^+)\left(b-\frac{s_1(\boldsymbol{\theta})-1}{N-1}c\right)-(1-\kappa_2x^+_{s_1(\boldsymbol{\theta})})\left(\frac{s_0(\boldsymbol{\theta})-1}{N-1}b-c\right)}{x^+_{s_1(\boldsymbol{\theta})}-x^+_0}}.\qquad(95)$$

For (α^{f}-3), since $\frac{1+\kappa_1 x^+_{s_1(\boldsymbol{\theta})}}{1+\kappa_1 x^+_0} < 1+\kappa_1$, the RHS is always smaller than 0. Hence, we have

$$(\alpha^{\mathrm{f}}\text{-}3) \Leftrightarrow \delta \leq \frac{\frac{(1+\kappa_1 x_0^+)v^1-(1+\kappa_1 x^+_{s_1(\boldsymbol{\theta})})v^0}{x^+_{s_1(\boldsymbol{\theta})}-x_0^+} - \frac{(1+\kappa_1 x_0^+)\left(b-\frac{s_1(\boldsymbol{\theta})-1}{N-1}c\right)-(1+\kappa_1 x^+_{s_1(\boldsymbol{\theta})})\left(\frac{s_0(\boldsymbol{\theta})-1}{N-1}b-c\right)}{x^+_{s_1(\boldsymbol{\theta})}-x_0^+}}{z_3 - \frac{(1+\kappa_1 x_0^+)\left(b-\frac{s_1(\boldsymbol{\theta})-1}{N-1}c\right)-(1+\kappa_1 x^+_{s_1(\boldsymbol{\theta})})\left(\frac{s_0(\boldsymbol{\theta})-1}{N-1}b-c\right)}{x^+_{s_1(\boldsymbol{\theta})}-x_0^+}}. \tag{96}$$

We briefly summarize what requirements on δ we have obtained now. To make the continuation payoff profile in the self-generating under the decomposition of α^{a}, we have one upper bound on δ resulting from (α^{a}-2) and one lower bound on δ resulting from (α^{a}-3). To make the continuation payoff profile in the self-generating under the decomposition of α^{f}, we have two upper bounds on δ resulting from (α^{f}-2) and (α^{f}-3), and one lower bound on δ resulting from (α^{f}-1). First, we want to eliminate the upper bounds, namely, make the upper bounds larger than 1, such that δ can be arbitrarily close to 1.

To eliminate the following upper bound resulting from (α^{a}-2),

$$\delta \leq \frac{\frac{(1-\kappa_2 x_0^+)v^1-(1-\kappa_2 x_1^+)v^0}{x_1^+-x_0^+} - \kappa_2\cdot(b-c)}{z_2-\kappa_2\cdot(b-c)}, \tag{97}$$

we need to have (since $z_2-\kappa_2\cdot(b-c)<0$)

$$\frac{(1-\kappa_2 x_0^+)v^1-(1-\kappa_2 x_1^+)v^0}{x_1^+-x_0^+} \leq z_2,\ \ \forall v^1, v^0. \tag{98}$$

The LHS of this inequality is maximized when $v^0 = \frac{z_2-z_3}{\kappa_1+\kappa_2}$ and $v^1 = v^0 + \frac{\kappa_1 z_2+\kappa_2 z_3}{\kappa_1+\kappa_2}$. Hence, the inequality is satisfied if

$$\frac{(1-\kappa_2 x_0^+)\left(\frac{z_2-z_3}{\kappa_1+\kappa_2}+\frac{\kappa_1 z_2+\kappa_2 z_3}{\kappa_1+\kappa_2}\right)-(1-\kappa_2 x_1^+)\frac{z_2-z_3}{\kappa_1+\kappa_2}}{x_1^+-x_0^+} \leq z_2 \tag{99}$$

$$\Leftrightarrow \left(\frac{1-x_1^+-x_0^+(\kappa_2-1)}{x_1^+-x_0^+}\frac{\kappa_1}{\kappa_1+\kappa_2}\right)z_2 \leq -\frac{1-x_1^+-x_0^+(\kappa_2-1)}{x_1^+-x_0^+}\frac{\kappa_2}{\kappa_1+\kappa_2}z_3. \tag{100}$$

Since $x_0^+ < \frac{1-\beta_1^+}{1-\kappa_2} < \frac{1-x_1^+}{1-\kappa_2}$, we have

$$z_2 \leq -\frac{\kappa_2}{\kappa_1}z_3. \tag{101}$$

To eliminate the following upper bound resulting from (α^{f}-2),

$$\delta \leq \frac{\frac{(1-\kappa_2 x_0^+)v^1-(1-\kappa_2 x^+_{s_1(\boldsymbol{\theta})})v^0}{x^+_{s_1(\boldsymbol{\theta})}-x_0^+} - \frac{(1-\kappa_2 x_0^+)\left(b-\frac{s_1(\boldsymbol{\theta})-1}{N-1}c\right)-(1-\kappa_2 x^+_{s_1(\boldsymbol{\theta})})\left(\frac{s_0(\boldsymbol{\theta})-1}{N-1}b-c\right)}{x^+_{s_1(\boldsymbol{\theta})}-x_0^+}}{z_2 - \frac{(1-\kappa_2 x_0^+)\left(b-\frac{s_1(\boldsymbol{\theta})-1}{N-1}c\right)-(1-\kappa_2 x^+_{s_1(\boldsymbol{\theta})})\left(\frac{s_0(\boldsymbol{\theta})-1}{N-1}b-c\right)}{x^+_{s_1(\boldsymbol{\theta})}-x_0^+}}, \tag{102}$$

we need to have (since $z_2 - \frac{(1-\kappa_2 x_0^+)(b-\frac{s_1(\boldsymbol{\theta})-1}{N-1}c)-(1-\kappa_2 x^+_{s_1(\boldsymbol{\theta})})(\frac{s_0(\boldsymbol{\theta})-1}{N-1}b-c)}{x^+_{s_1(\boldsymbol{\theta})}-x_0^+} < 0$)

$$\frac{(1-\kappa_2 x_0^+)v^1-(1-\kappa_2 x^+_{s_1(\boldsymbol{\theta})})v^0}{x^+_{s_1(\boldsymbol{\theta})}-x_0^+} \leq z_2,\ \ \forall v^1, v^0. \tag{103}$$

Similarly, the LHS of this inequality is maximized when $v^0 = \frac{z_2 - z_3}{\kappa_1 + \kappa_2}$ and $v^1 = v^0 + \frac{\kappa_1 z_2 + \kappa_2 z_3}{\kappa_1 + \kappa_2}$. Hence, the inequality is satisfied if

$$\frac{(1-\kappa_2 x_0^+)\left(\frac{z_2-z_3}{\kappa_1+\kappa_2} + \frac{\kappa_1 z_2 + \kappa_2 z_3}{\kappa_1+\kappa_2}\right) - (1-\kappa_2 x_{s_1(\boldsymbol{\theta})}^+)\frac{z_2-z_3}{\kappa_1+\kappa_2}}{x_{s_1(\boldsymbol{\theta})}^+ - x_0^+} \le z_2 \tag{104}$$

$$\Leftrightarrow \left(\frac{1 - x_{s_1(\boldsymbol{\theta})}^+ - x_0^+(\kappa_2 - 1)}{x_{s_1(\boldsymbol{\theta})}^+ - x_0^+}\frac{\kappa_1}{\kappa_1+\kappa_2}\right) z_2 \le -\frac{1 - x_{s_1(\boldsymbol{\theta})}^+ - x_0^+(\kappa_2 - 1)}{x_{s_1(\boldsymbol{\theta})}^+ - x_0^+}\frac{\kappa_2}{\kappa_1+\kappa_2} z_3. \tag{105}$$

Since $x_0^+ < \frac{1-\beta_1^+}{1-\kappa_2} < \frac{1-x_{s_1(\boldsymbol{\theta})}^+}{1-\kappa_2}$, we have

$$z_2 \le -\frac{\kappa_2}{\kappa_1} z_3. \tag{106}$$

To eliminate the following upper bound resulting from (α^{f}-3),

$$\delta \le \frac{\frac{(1+\kappa_1 x_0^+)v^1 - (1+\kappa_1 x_{s_1(\boldsymbol{\theta})}^+)v^0}{x_{s_1(\boldsymbol{\theta})}^+ - x_0^+} - \frac{(1+\kappa_1 x_0^+)\left(b - \frac{s_1(\boldsymbol{\theta})-1}{N-1}c\right) - (1+\kappa_1 x_{s_1(\boldsymbol{\theta})}^+)\left(\frac{s_0(\boldsymbol{\theta})-1}{N-1}b - c\right)}{x_{s_1(\boldsymbol{\theta})}^+ - x_0^+}}{z_3 - \frac{(1+\kappa_1 x_0^+)\left(b - \frac{s_1(\boldsymbol{\theta})-1}{N-1}c\right) - (1+\kappa_1 x_{s_1(\boldsymbol{\theta})}^+)\left(\frac{s_0(\boldsymbol{\theta})-1}{N-1}b - c\right)}{x_{s_1(\boldsymbol{\theta})}^+ - x_0^+}}, \tag{107}$$

we need to have (since $z_3 - \frac{(1+\kappa_1 x_0^+)(b - \frac{s_1(\boldsymbol{\theta})-1}{N-1}c) - (1+\kappa_1 x_{s_1(\boldsymbol{\theta})}^+)(\frac{s_0(\boldsymbol{\theta})-1}{N-1}b - c)}{x_{s_1(\boldsymbol{\theta})}^+ - x_0^+} < 0$)

$$\frac{(1+\kappa_1 x_0^+)v^1 - (1+\kappa_1 x_{s_1(\boldsymbol{\theta})}^+)v^0}{x_{s_1(\boldsymbol{\theta})}^+ - x_0^+} \le z_3, \ \forall v^1, v^0. \tag{108}$$

Again, the LHS of this inequality is maximized when $v^0 = \frac{z_2 - z_3}{\kappa_1 + \kappa_2}$ and $v^1 = v^0 + \frac{\kappa_1 z_2 + \kappa_2 z_3}{\kappa_1 + \kappa_2}$. Hence, the inequality is satisfied if

$$\frac{(1+\kappa_1 x_0^+)\left(\frac{z_2-z_3}{\kappa_1+\kappa_2} + \frac{\kappa_1 z_2 + \kappa_2 z_3}{\kappa_1+\kappa_2}\right) - (1+\kappa_1 x_{s_1(\boldsymbol{\theta})}^+)\frac{z_2-z_3}{\kappa_1+\kappa_2}}{x_{s_1(\boldsymbol{\theta})}^+ - x_0^+} \le z_3 \tag{109}$$

$$\Leftrightarrow \left(\frac{1 - x_{s_1(\boldsymbol{\theta})}^+ + x_0^+(\kappa_1 + 1)}{x_{s_1(\boldsymbol{\theta})}^+ - x_0^+}\frac{\kappa_1}{\kappa_1+\kappa_2}\right) z_2 \le -\frac{1 - x_{s_1(\boldsymbol{\theta})}^+ + x_0^+(\kappa_1 + 1)}{x_{s_1(\boldsymbol{\theta})}^+ - x_0^+}\frac{\kappa_2}{\kappa_1+\kappa_2} z_3. \tag{110}$$

Since $1 - x_{s_1(\boldsymbol{\theta})}^+ + x_0^+(\kappa_1 + 1) > 0$, we have

$$z_2 \le -\frac{\kappa_2}{\kappa_1} z_3. \tag{111}$$

In summary, to eliminate the upper bounds on δ, we only need to have $z_2 \le -\frac{\kappa_2}{\kappa_1} z_3$, which is satisfied since we define $z_3 \triangleq -\frac{\kappa_1}{\kappa_2} z_2$.

Now we derive the analytical lower bound on δ based on the lower bounds resulting from (α^{a}-3) and (α^{f}-1):

$$(\alpha^{\mathrm{a}}\text{-}3) \Leftrightarrow \delta \ge \frac{\frac{(1+\kappa_1 x_0^+)v^1 - (1+\kappa_1 x_1^+)v^0}{x_1^+ - x_0^+} + \kappa_1 \cdot (b - c)}{z_3 + \kappa_1 \cdot (b - c)}, \tag{112}$$

and

$$\delta \geq \frac{v^1 - v^0 - \left(\frac{s_1(\boldsymbol{\theta})}{N-1}b + \frac{s_0(\boldsymbol{\theta})}{N-1}c\right)}{(\epsilon^0 - \epsilon^1)(x^+_{s_1(\boldsymbol{\theta})} - x^+_0) - \left(\frac{s_1(\boldsymbol{\theta})}{N-1}b + \frac{s_0(\boldsymbol{\theta})}{N-1}c\right)},\ \forall \boldsymbol{\theta}\ s.t.\ \frac{s_1(\boldsymbol{\theta})}{N-1}b + \frac{s_0(\boldsymbol{\theta})}{N-1}c \geq v^1 - v^0. \quad (113)$$

We define an intermediate lower bound based on the latter inequality along with the inequalities resulting from the incentive compatibility constraints.

$$\underline{\delta}' = \max\left\{ \max_{\substack{s_1\in\{1,\ldots,N-1\}:\\ \frac{s_1}{N-1}b+\frac{N-s_1}{N-1}c>\epsilon_0-\epsilon_1}} \frac{\epsilon_0 - \epsilon_1 - \left(\frac{s_1}{N-1}b + \frac{N-s_1}{N-1}c\right)}{(\epsilon_0-\epsilon_1)\left(\frac{N-s_1}{N-1}\beta^+_1 + \frac{s_1-1}{N-1}x^+_1\right) - \left(\frac{s_1}{N-1}b + \frac{N-s_1}{N-1}c\right)}, \max_{\theta\in\{0,1\}} \frac{c}{c + (1-2\varepsilon)(\beta^+_\theta - (1-\beta^-_\theta))(\epsilon_0-\epsilon_1)} \right\}. \quad (114)$$

Then the lower bound can be written as $\underline{\delta} = \max\left\{\underline{\delta}', \underline{\delta}''\right\}$, where $\underline{\delta}''$ is the lower bound that we will derive for the case when the users have the same rating. If the payoffs v^1 and v^0 satisfy the constraint resulting from (α^{a}-3), namely, satisfying

$$\frac{(1+\kappa_1 x^+_0)v^1 - (1+\kappa_1 x^+_1)v^0}{x^+_1 - x^+_0} \leq \underline{\delta} z_3 - (1-\underline{\delta})\kappa_1 \cdot (b-c), \quad (115)$$

then we use α^{a} to decompose v^1 and v^0. Otherwise, we use α^{f} to decompose v^1 and v^0.

B.3. When the Users Have the Same Rating

Now we derive the conditions under which any payoff profile in $\mathcal{W}^{\mathbf{1}_N}$ and $\mathcal{W}^{\mathbf{0}_N}$ can be decomposed.

If all the users have rating 1, namely, $\boldsymbol{\theta} = \mathbf{1}_N$, to decompose $\boldsymbol{v} \in \mathcal{W}^{\mathbf{1}_N}$, we need to find a recommended plan α_0 and a simple continuation payoff function $\boldsymbol{\gamma}$ such that for all $i \in \mathcal{N}$ and for all $\alpha_i \in A$,

$$\begin{aligned} v_i &= (1-\delta)u_i(\boldsymbol{\theta}, \alpha_0, \alpha_0 \cdot \mathbf{1}_N) + \delta \sum_{\boldsymbol{\theta}'} \gamma_i(\boldsymbol{\theta}')q(\boldsymbol{\theta}'|\boldsymbol{\theta}, \alpha_0, \alpha_0 \cdot \mathbf{1}_N) \quad (116)\\ &\geq (1-\delta)u_i(\boldsymbol{\theta}, \alpha_0, \alpha_i, \alpha_0 \cdot \mathbf{1}_{N-1}) + \delta \sum_{\boldsymbol{\theta}'} \gamma_i(\boldsymbol{\theta}')q(\boldsymbol{\theta}'|\boldsymbol{\theta}, \alpha_0, \alpha_i, \alpha_0 \cdot \mathbf{1}_{N-1}). \end{aligned}$$

When all the users have the same rating, the altruistic plan α^{a} is equivalent to the fair plan α^{f}. Hence, we use the altruistic plan and the selfish plan to decompose the payoff profiles.

If we use the altruistic plan α^{a} to decompose a payoff profile $\boldsymbol{v}$, we have

$$v^1 = (1-\delta)(b-c) + \delta(x^+_1\gamma^1 + (1-x^+_1)\gamma^0), \quad (117)$$

and the incentive compatibility constraint

$$(1-2\varepsilon)\left[\beta^+_1 - (1-\beta^-_1)\right](\gamma^1 - \gamma^0) \geq \frac{1-\delta}{\delta}c. \quad (118)$$

Setting $\gamma^1 = \gamma^0 + \frac{1-\delta}{\delta}\frac{c}{(1-2\varepsilon)[\beta_1^+ - (1-\beta_1^-)]}$ and noticing that $\gamma^0 \in [\frac{(1+\kappa_1)(\epsilon_0-\epsilon_1)-z_3}{\kappa_1}, \frac{\kappa_1 z_2+(\kappa_2-1)z_3}{\kappa_1+\kappa_2}]$, we get a lower bound on v^1 that can be decomposed by α^{a}.

$$v^1 = (1-\delta)(b-c) + \delta\left(\gamma^0 + x_1^+ \frac{1-\delta}{\delta}\frac{c}{(1-2\varepsilon)\left[\beta_1^+ - (1-\beta_1^-)\right]}\right) \tag{119}$$

$$\geq (1-\delta)\left(b-c+c\frac{x_1^+}{(1-2\varepsilon)\left[\beta_1^+ - (1-\beta_1^-)\right]}\right) + \delta\frac{(1+\kappa_1)(\epsilon_0-\epsilon_1)-z_3}{\kappa_1}. \tag{120}$$

If we use the selfish plan α^{s} to decompose a payoff profile $\boldsymbol{v}$, we have

$$v^1 = \delta(x_1^+\gamma^1 + (1-x_1^+)\gamma^0). \tag{121}$$

Since the selfish plan is NE of the stage game, the incentive compatibility constraint is satisfied as long as we set $\gamma^1 = \gamma^0$. Hence, we have $v^1 = \delta\gamma^0$. Again, noticing that $\gamma^0 \in [\frac{(1+\kappa_1)(\epsilon_0-\epsilon_1)-z_3}{\kappa_1}, \frac{\kappa_1 z_2+(\kappa_2-1)z_3}{\kappa_1+\kappa_2}]$, we get an upper bound on v^1 that can be decomposed by α^{s}

$$v^1 = \delta\gamma^0 \leq \delta\frac{\kappa_1 z_2 + (\kappa_2-1)z_3}{\kappa_1+\kappa_2}. \tag{122}$$

In order to decompose any payoff profile $\boldsymbol{v} \in \mathcal{W}^{\mathbf{1}_N}$, the lower bound on v^1 that can be decomposed by α^{a} must be smaller than the upper bound on v^1 that can be decomposed by α^{s}, which leads to

$$(1-\delta)\left(b-c+c\frac{x_1^+}{(1-2\varepsilon)\left[\beta_1^+ - (1-\beta_1^-)\right]}\right) + \delta\frac{(1+\kappa_1)(\epsilon_0-\epsilon_1)-z_3}{\kappa_1} \leq \delta\frac{\kappa_1 z_2+(\kappa_2-1)z_3}{\kappa_1+\kappa_2}$$

$$\Rightarrow \delta \geq \frac{b-c+c\frac{x_1^+}{(1-2\varepsilon)\left[\beta_1^+ - (1-\beta_1^-)\right]}}{b-c+c\frac{x_1^+}{(1-2\varepsilon)\left[\beta_1^+ - (1-\beta_1^-)\right]} + \frac{\kappa_1 z_2+(\kappa_2-1)z_3}{\kappa_1+\kappa_2} - \frac{(1+\kappa_1)(\epsilon_0-\epsilon_1)-z_3}{\kappa_1}}. \tag{123}$$

Finally, following the same procedure, we derive the lower bound on δ when all the users have rating 0, namely, $\boldsymbol{\theta} = \mathbf{0}_N$. Similarly, in this case, the altruistic plan α^{a} is equivalent to the fair plan α^{f}. Hence, we use the altruistic plan and the selfish plan to decompose the payoff profiles.

If we use the altruistic plan α^{a} to decompose a payoff profile $\boldsymbol{v}$, we have

$$v^0 = (1-\delta)(b-c) + \delta(x_0^+\gamma^1 + (1-x_0^+)\gamma^0), \tag{124}$$

and the incentive compatibility constraint

$$(1-2\varepsilon)\left[\beta_0^+ - (1-\beta_0^-)\right](\gamma^1-\gamma^0) \geq \frac{1-\delta}{\delta}c. \tag{125}$$

If we use the selfish plan α^{s} to decompose a payoff profile $\boldsymbol{v}$, we have

$$v^1 = \delta(x_0^+\gamma^1 + (1-x_0^+)\gamma^0). \tag{126}$$

Note that when $\boldsymbol{\theta} = \mathbf{0}_N$, if we substitute β_0^+, β_0^-, x_0^- with β_1^+, β_1^-, x_1^-, respectively, the decomposability constraints become the same as those when $\boldsymbol{\theta} = \mathbf{1}_N$. Hence, we derive a similar lower bound on δ.

$$\delta \geq \frac{b-c+c\frac{x_0^+}{(1-2\varepsilon)\left[\beta_0^+ - (1-\beta_0^-)\right]}}{b-c+c\frac{x_0^+}{(1-2\varepsilon)\left[\beta_0^+ - (1-\beta_0^-)\right]} + \frac{\kappa_1 z_2+(\kappa_2-1)z_3}{\kappa_1+\kappa_2} - \frac{(1+\kappa_1)(\epsilon_0-\epsilon_1)-z_3}{\kappa_1}}. \tag{127}$$

Finally, we can obtain the lower bound on δ when the users have the same rating as

$$\underline{\delta}'' = \max_{\theta\in\{0,1\}} \frac{b-c+c\frac{x_\theta^+}{(1-2\varepsilon)\left[\beta_\theta^+-(1-\beta_\theta^-)\right]}}{b-c+c\frac{x_\theta^+}{(1-2\varepsilon)\left[\beta_\theta^+-(1-\beta_\theta^-)\right]}+\frac{\kappa_1 z_2+(\kappa_2-1)z_3}{\kappa_1+\kappa_2}-\frac{(1+\kappa_1)(\epsilon_0-\epsilon_1)-z_3}{\kappa_1}}. \tag{128}$$

Together with the lower bound $\underline{\delta}'$ derived for the case when the users have different ratings, we can get the lower bound $\underline{\delta}$ specified in Condition 3 of Theorem 5.2.

C. COMPLETE DESCRIPTION OF THE ALGORITHM

Table IV presents the complete description of the algorithm for constructing the equilibrium strategy by the rating mechanism.

Table IV. Algorithm of Constructing the Equilibrium Strategy by the Rating Mechanism

Require: b, c, ε, ξ; $\tau(\varepsilon)$, $\delta \geq \underline{\delta}(\varepsilon,\xi)$; $\boldsymbol{\theta}^0$	*(inputs to the algorithm)*
Initialization: $t=0$, $\epsilon_0=\xi$, $\epsilon_1=\epsilon_0/(1+\frac{\kappa_2}{\kappa_1})$, $v^\theta=b-c-\epsilon_\theta$, $\boldsymbol{\theta}=\boldsymbol{\theta}^0$.	*(set the target payoffs)*
repeat	
if $s_1(\boldsymbol{\theta})=0$ **then**	
if $v^0 \geq (1-\delta)\left[b-c+\frac{(1-\varepsilon)\beta_0^++\varepsilon(1-\beta_0^-)}{(1-2\varepsilon)(\beta_0^+-(1-\beta_0^-))}c\right]+\delta\frac{\epsilon_0-\epsilon_1-z_3}{\kappa_1}$ **then**	
$\alpha_0^t=\alpha^{\mathrm{a}}$	*(determine the recommended plan)*
$v^0 \leftarrow \frac{v^0}{\delta}-\frac{1-\delta}{\delta}\left[b-c+\frac{(1-\varepsilon)\beta_0^++\varepsilon(1-\beta_0^-)}{(1-2\varepsilon)(\beta_0^+-(1-\beta_0^-))}c\right]$, $v^1 \leftarrow v^0+\frac{1-\delta}{\delta}\left[\frac{1}{(1-2\varepsilon)(\beta_0^+-(1-\beta_0^-))}c\right]$	*(update the continuation payoff)*
else	
$\alpha_0^t=\alpha^{\mathrm{s}}$	*(determine the recommended plan)*
$v^0 \leftarrow \frac{v^0}{\delta}$, $v^1 \leftarrow v^0$	*(update the continuation payoff)*
end	
elseif $s_1(\boldsymbol{\theta})=N$ **then**	
if $v^1 \geq (1-\delta)\left[b-c+\frac{(1-\varepsilon)\beta_1^++\varepsilon(1-\beta_1^-)}{(1-2\varepsilon)(\beta_1^+-(1-\beta_1^-))}c\right]+\delta\frac{\epsilon_0-\epsilon_1-z_3}{\kappa_1}$ **then**	
$\alpha_0^t=\alpha^{\mathrm{a}}$	*(determine the recommended plan)*
$v^1 \leftarrow \frac{v^1}{\delta}-\frac{1-\delta}{\delta}\left[b-c+\frac{(1-\varepsilon)\beta_1^++\varepsilon(1-\beta_1^-)}{(1-2\varepsilon)(\beta_1^+-(1-\beta_1^-))}c\right]$, $v^0 \leftarrow v^1-\frac{1-\delta}{\delta}\left[\frac{1}{(1-2\varepsilon)(\beta_1^+-(1-\beta_1^-))}c\right]$	*(update the continuation payoff)*
else	
$\alpha_0^t=\alpha^{\mathrm{s}}$	*(determine the recommended plan)*
$v^1 \leftarrow \frac{v^1}{\delta}$, $v^0 \leftarrow v^1$	*(update the continuation payoff)*
end	
else	
if $\frac{1+\kappa_1 x_0^+}{x_1^+-x_0^+}v^1-\frac{1+\kappa_1 x_1^+}{x_1^+-x_0^+}v^0 \leq \delta z_3-(1-\delta)\kappa_1(b-c)$ **then**	
$\alpha_0^t=\alpha^{\mathrm{a}}$	*(determine the recommended plan)*
$v^{1\prime} \leftarrow \frac{1}{\delta}\frac{(1-x_0^+)v^1-(1-x_1^+)v^0}{x_1^+-x_0^+}-\frac{1-\delta}{\delta}(b-c)$, $v^{0\prime} \leftarrow \frac{1}{\delta}\frac{x_1^+v^0-x_0^+v^1}{x_1^+-x_0^+}-\frac{1-\delta}{\delta}(b-c)$	*(update the continuation payoff)*
$v^1 \leftarrow v^{1\prime}$, $v^0 \leftarrow v^{0\prime}$	
else	
$\alpha_0^t=\alpha^{\mathrm{f}}$	*(determine the recommended plan)*
$v^{1\prime} \leftarrow \frac{1}{\delta}\frac{(1-x_0^+)v^1-(1-x_{s_1(\boldsymbol{\theta})}^+)v^0}{x_{s_1(\boldsymbol{\theta})}^+-x_0^+}-\frac{1-\delta}{\delta}\frac{(b-\frac{s_1(\boldsymbol{\theta})-1}{N-1}c)(1-x_0^+)-(\frac{s_0(\boldsymbol{\theta})-1}{N-1}b-c)(1-x_{s_1(\boldsymbol{\theta})}^+)}{x_{s_1(\boldsymbol{\theta})}^+-x_0^+}$	*(update the continuation payoff)*
$v^{1\prime} \leftarrow \frac{1}{\delta}\frac{x_0^+v^1-x_{s_1(\boldsymbol{\theta})}^+v^0}{x_0^+-x_{s_1(\boldsymbol{\theta})}^+}-\frac{1-\delta}{\delta}\frac{(b-\frac{s_1(\boldsymbol{\theta})-1}{N-1}c)x_0^+-(\frac{s_0(\boldsymbol{\theta})-1}{N-1}b-c)x_{s_1(\boldsymbol{\theta})}^+}{x_0^+-x_{s_1(\boldsymbol{\theta})}^+}$	
$v^1 \leftarrow v^{1\prime}$, $v^0 \leftarrow v^{0\prime}$	
end	
end	
$t \leftarrow t+1$, determine the rating profile $\boldsymbol{\theta}^t$, set $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta}^t$	
until $\varnothing$	