Solution to EE 617 Mid-Term Exam, Fall 2017

November 2, 2017

1. (4 points) *Convex sets.*

   (a) (2 points) Consider the set

$$\left\{a \in \mathbb{R}^k \mid p(0) = 1,\ |p(t)| \leq 1 \text{ for } \alpha \leq t \leq \beta\right\},$$

where $p(t) = a_1 + a_2 t + \cdots + a_k t^{k-1}$. Is this set convex?

Solution:
The set can be rewritten as

$$\begin{aligned}
& \left\{a \in \mathbb{R}^k \mid p(0) = 1,\ |p(t)| \leq 1 \text{ for } \alpha \leq t \leq \beta\right\} \\
= \ & \left\{a \in \mathbb{R}^k \mid p(0) = 1\right\} \cap \left(\bigcap_{t \in [\alpha,\beta]} \left\{a \in \mathbb{R}^k \mid |p(t)| \leq 1\right\} \right) \\
= \ & \left\{a \in \mathbb{R}^k \mid a_1 = 1\right\} \cap \left(\bigcap_{t \in [\alpha,\beta]} \left\{a \in \mathbb{R}^k \mid -1 \leq a_1 + a_2 t + \cdots + a_k t^{k-1} \leq 1\right\} \right).
\end{aligned}$$

The set is the intersection of a hyperplane and infinitely many halfspaces. Therefore, it is convex.

   (b) (2 points) The polar of a set $C \subseteq \mathbb{R}^n$ is defined as

$$C^\circ = \left\{y \in \mathbb{R}^n \mid y^T x \leq 1 \text{ for all } x \in C\right\}.$$

Is the polar C° of a (possibly nonconvex) set C convex?

Solution:
The set can be rewritten as

$$C^\circ = \bigcap_{x \in C} \left\{y \in \mathbb{R}^n \mid y^T x \leq 1\right\}.$$

The set is the intersection of halfspaces. Therefore, it is convex, regardless of whether C is convex or not.

2. (4 points) *Convex functions.*

(a) (2 points) The logarithmic barrier for the second-order cone constraint $\|x\|_2 \leq t$ is

$$f(x,t) = -\log\left(t^2 - x^T x\right), \text{ with } \mathbf{dom} f = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\|_2 < t\}.$$

Show that the function $f(x,t)$ is convex in (x,t). (*Hint:* use the convexity of $x^T x/t$.)

Solution:
The domain of f is a second-order cone, and hence is a convex set.
We can rewrite the function as

$$f(x,t) = -\log\left[t \cdot \left(t - x^T x/t\right)\right] = -\log t - \log\left(t - x^T x/t\right).$$

Clearly, the first term $-\log t$ is convex.
For the second term, note that $x^T x/t$ is convex, and therefore $t - x^T x/t$ is concave. Hence, $\log\left(t - x^T x/t\right)$ is concave, and $-\log\left(t - x^T x/t\right)$ is convex.
In summary, the function $f(x,t)$ is the sum of two convex functions. This concludes the proof.

(b) (2 points) The directional derivative of a function $g : \mathbb{R}^n \to \mathbb{R}$ is

$$f(x) = \inf_{\alpha > 0} \frac{g(y + \alpha x) - g(y)}{\alpha}.$$

Show that $f : \mathbb{R}^n \to \mathbb{R}$ is convex if g is convex.

Solution:
The domain of f is $\mathbb{R}^n$, and hence is a convex set.
We show the convexity of f by definition. For any $x, z \in \mathbb{R}^n$ and $\theta \in [0,1]$, we have

$$\begin{aligned} f\left(\theta x + (1-\theta)z\right) &= \inf_{\alpha > 0} \frac{g\left[y + \alpha\left(\theta x + (1-\theta)z\right)\right] - g(y)}{\alpha} \\ &= \inf_{\alpha > 0} \frac{g\left[\theta\left(y + \alpha x\right) + (1-\theta)\left(y + \alpha z\right)\right] - g(y)}{\alpha} \end{aligned}$$

Using the convexity of g, we have

$$g\left[\theta\left(y + \alpha x\right) + (1-\theta)\left(y + \alpha z\right)\right] \leq \theta g\left(y + \alpha x\right) + (1-\theta) g\left(y + \alpha z\right)$$

Therefore, we have

$$\begin{aligned} f\left(\theta x + (1-\theta)z\right) &\leq \inf_{\alpha > 0} \frac{\theta g\left(y + \alpha x\right) + (1-\theta) g\left(y + \alpha z\right) - g(y)}{\alpha} \\ &= \inf_{\alpha > 0} \frac{\theta\left[g\left(y + \alpha x\right) - g(y)\right] + (1-\theta)\left[g\left(y + \alpha z\right) - g(y)\right]}{\alpha} \\ &= \inf_{\alpha > 0} \left[\theta \frac{g\left(y + \alpha x\right) - g(y)}{\alpha} + (1-\theta) \frac{g\left(y + \alpha z\right) - g(y)}{\alpha}\right]. \qquad (1) \end{aligned}$$

Now we need to show that

$$\inf_{\alpha>0} \frac{g(y+\alpha x)-g(y)}{\alpha} = \lim_{\alpha\downarrow 0} \frac{g(y+\alpha x)-g(y)}{\alpha},$$

which is true if $\frac{g(y+\alpha x)-g(y)}{\alpha}$ is nondecreasing in α. For any $0<\alpha_1<\alpha_2$, we have

$$\begin{aligned} g(y+\alpha_1 x) &= g\left[\left(1-\frac{\alpha_1}{\alpha_2}\right)y+\frac{\alpha_1}{\alpha_2}(y+\alpha_2 x)\right] \\ &\leq \left(1-\frac{\alpha_1}{\alpha_2}\right)g(y)+\frac{\alpha_1}{\alpha_2}g(y+\alpha_2 x), \end{aligned}$$

where the last inequality comes from the convexity of g. The above equation implies

$$\frac{g(y+\alpha_1 x)-g(y)}{\alpha_1} \leq \frac{g(y+\alpha_2 x)-g(y)}{\alpha_2}.$$

As a result, we have

$$\inf_{\alpha>0} \frac{g(y+\alpha x)-g(y)}{\alpha} = \lim_{\alpha\downarrow 0} \frac{g(y+\alpha x)-g(y)}{\alpha},$$

Finally, from (1), we have

$$\begin{aligned} f(\theta x+(1-\theta)z) &\leq \inf_{\alpha>0}\left[\theta\frac{g(y+\alpha x)-g(y)}{\alpha}+(1-\theta)\frac{g(y+\alpha z)-g(y)}{\alpha}\right] \\ &= \lim_{\alpha\downarrow 0}\left[\theta\frac{g(y+\alpha x)-g(y)}{\alpha}+(1-\theta)\frac{g(y+\alpha z)-g(y)}{\alpha}\right] \\ &= \theta\lim_{\alpha\downarrow 0}\frac{g(y+\alpha x)-g(y)}{\alpha}+(1-\theta)\lim_{\alpha\downarrow 0}\frac{g(y+\alpha z)-g(y)}{\alpha} \\ &= \theta\inf_{\alpha>0}\frac{g(y+\alpha x)-g(y)}{\alpha}+(1-\theta)\inf_{\alpha>0}\frac{g(y+\alpha z)-g(y)}{\alpha} \\ &= \theta f(x)+(1-\theta)f(z). \end{aligned}$$

This concludes the proof that f is convex.

3. (4 points) *Convex optimization problems.*

(a) (2 points) Consider the following optimization problem:

$$\begin{array}{ll} \text{minimize} & \dfrac{\max_{i=1,\dots,m}\left(a_i^T x + b_i\right)}{\min_{i=1,\dots,p}\left(c_i^T x + d_i\right)} \\ \text{subject to} & Fx \le g, \end{array}$$

with variables $x \in \mathbb{R}^n$. Assume that $c_i^T x + d_i > 0$ for all x that satisfy $Fx \le g$. Give the best convex reformulation of the problem. By "best", we mean LP is better than SOCP, SOCP is better than SDP, and SDP is better than general convex program.

Solution:
The objective function is the ratio of a convex function to a concave function. Therefore, the problem is a convex-concave fractional problem in Exercise 4.7 (c) of the book. We will use the same approach here.
The convex formulation is

$$\begin{array}{ll} \text{minimize} & g_0(y,t) \\ \text{subject to} & g_i(y,t) \le 0,\ i = 1,\dots,m \\ & \tilde{h}(y,t) \le -1, \end{array}$$

where g_0 is the perspective of $f_0(x) = \max_{i=1,\dots,m}\left(a_i^T x + b_i\right)$, $\tilde{h}$ is the perspective of $-h(x) = -\min_{i=1,\dots,p}\left(c_i^T x + d_i\right)$, and g_i is the perspective of $f_i(x) = f_i^T x - g_i$, where f_i^T is the ith row of matrix F and g_i is the ith element of vector g. Next, we compute the above functions explicitly.
The objective function is $g_0 : \mathbb{R}^n \times \mathbb{R}_{++} \to \mathbb{R}$, which can be calculated as

$$g_0(y,t) = t f_0(y/t) = t \cdot \max_{i=1,\dots,m}\left(a_i^T (y/t) + b_i\right) = \max_{i=1,\dots,m}\left(a_i^T y + b_i t\right).$$

Similarly, we can compute $\tilde{h} : \mathbb{R}^n \times \mathbb{R}_{++} \to \mathbb{R}$ as

$$\tilde{h}(y,t) = -t h(y/t) = -t \cdot \min_{i=1,\dots,p}\left(c_i^T (y/t) + d_i\right) = -\min_{i=1,\dots,p}\left(c_i^T y + d_i t\right).$$

Finally, we have $g_i : \mathbb{R}^n \times \mathbb{R}_{++} \to \mathbb{R}$ as

$$g_i(y,t) = t f_i(y/t) = t \cdot \left(f_i^T (y/t) - g_i\right) = f_i^T y - g_i t.$$

Therefore, the original problem can be reformulated as

$$\begin{array}{ll} \text{minimize} & \displaystyle\max_{i=1,\dots,m}\left(a_i^T y + b_i t\right) \\ \text{subject to} & f_i^T y - g_i t \le 0,\ \forall i = 1,\dots,m \\ & \displaystyle -\min_{i=1,\dots,p}\left(c_i^T y + d_i t\right) \le -1, \end{array}$$

which is equivalent to the LP

$$\begin{array}{ll} \text{minimize} & z \\ \text{subject to} & a_i^T y + b_i t \le z,\ i = 1,\dots,m \\ & Fy - tg \le 0 \\ & c_i^T y + d_i t \ge 1,\ i = 1,\dots,p. \end{array}$$

(b) (2 points) Consider the following complex least ℓ_∞-norm problem:

$$\begin{array}{ll} \text{minimize} & \|x\|_\infty \\ \text{subject to} & Ax = b, \end{array}$$

with *complex* variables $x \in \mathbb{C}^n$ and *complex* parameters $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$. Note that

$$\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|.$$

Formulate the complex least ℓ_∞-norm problem as a problem with real parameters and real variables. Moreover, give the best convex reformulation of the new problem. By "best", we mean LP is better than SOCP, SOCP is better than SDP, and SDP is better than general convex program. (*Hint:* use $z = (\mathbf{Re}(x), \mathbf{Im}(x)) \in \mathbb{R}^{2n}$ as the variable.)

Solution:
We first write the objective function and constraints in real variables $z = (\mathbf{Re}(x), \mathbf{Im}(x)) \in \mathbb{R}^{2n}$. The objective function is

$$\|x\|_\infty = \max_{i=1,\ldots,n} |x_i| = \max_{i=1,\ldots,n} \left(\mathbf{Re}(x_i)^2 + \mathbf{Im}(x_i)^2\right)^{1/2}.$$

Each term $\left(\mathbf{Re}(x_i)^2 + \mathbf{Im}(x_i)^2\right)^{1/2}$ can be written compactly as

$$\left(\mathbf{Re}(x_i)^2 + \mathbf{Im}(x_i)^2\right)^{1/2} = \left\| \begin{bmatrix} \mathbf{Re}(x_i) \\ \mathbf{Im}(x_i) \end{bmatrix} \right\|_2^2 = \left\| C_i \begin{bmatrix} \mathbf{Re}(x) \\ \mathbf{Im}(x) \end{bmatrix} \right\|_2^2 = \|C_i z\|_2^2,$$

where $C_i \in \mathbb{R}^{2\times 2n}$ is a matrix such that

$$\begin{bmatrix} \mathbf{Re}(x_i) \\ \mathbf{Im}(x_i) \end{bmatrix} = C_i \begin{bmatrix} \mathbf{Re}(x) \\ \mathbf{Im}(x) \end{bmatrix}.$$

In particular, the matrix C_i has zero elements except $(C_i)_{1,i} = 1$ and $(C_i)_{2,n+i} = 1$.
Using $\mathbf{j} = \sqrt{-1}$, the linear equality can be rewritten as

$$\begin{aligned} Ax &= [\mathbf{Re}(A) + \mathbf{j}\mathbf{Im}(A)]\,[\mathbf{Re}(x) + \mathbf{j}\mathbf{Im}(x)] \\ &= [\mathbf{Re}(A)\mathbf{Re}(x) - \mathbf{Im}(A)\mathbf{Im}(x)] + \mathbf{j}\,[\mathbf{Re}(A)\mathbf{Im}(x) + \mathbf{Im}(A)\mathbf{Re}(x)] \\ &= \begin{bmatrix} \mathbf{Re}(A) & -\mathbf{Im}(A) \end{bmatrix} \begin{bmatrix} \mathbf{Re}(x) \\ \mathbf{Im}(x) \end{bmatrix} + \mathbf{j} \begin{bmatrix} \mathbf{Im}(A) & \mathbf{Re}(A) \end{bmatrix} \begin{bmatrix} \mathbf{Re}(x) \\ \mathbf{Im}(x) \end{bmatrix} \end{aligned}$$

Noting that $b = \mathbf{Re}(b) + \mathbf{j}\mathbf{Im}(b)$, the linear equality can be written compactly as

$$\begin{aligned} & \underbrace{\begin{bmatrix} \mathbf{Re}(A) & -\mathbf{Im}(A) \\ \mathbf{Im}(A) & \mathbf{Re}(A) \end{bmatrix}}_{\triangleq \tilde{A}} \begin{bmatrix} \mathbf{Re}(x) \\ \mathbf{Im}(x) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{Re}(b) \\ \mathbf{Im}(b) \end{bmatrix}}_{\triangleq \tilde{b}} \\ \Leftrightarrow \quad & \tilde{A}z = \tilde{b}. \end{aligned}$$

Therefore, the original problem in real variables can be written as

$$\begin{array}{ll} \text{minimize} & \max_{i=1,\ldots,n} \|C_i z\|_2^2 \\ \text{subject to} & \tilde{A}z = \tilde{b}, \end{array}$$

which is equivalent to the following SOCP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \|C_i z\|_2 \leq t,\ i = 1, \ldots, n \\ & \tilde{A} z = \tilde{b}. \end{array}$$

4. (4 points) *Dual problems.* Consider the problem of projecting a point $a \in \mathbb{R}^n$ on the unit ball in ℓ_1-norm:

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|x-a\|_2^2 \\ \text{subject to} & \|x\|_1 \leq 1, \end{array}$$

with variables $x \in \mathbb{R}^n$. Derive the dual problem. (*Hint:* use the additional variable $y = x - a$.)

Solution:
With the additional variable $y = x - a$, the problem is equivalent to

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|y\|_2^2 \\ \text{subject to} & \|x\|_1 \leq 1, \\ & y = x - a, \end{array}$$

with optimization variable $x, y \in \mathbb{R}^n$.

The Lagrangian is

$$L(x, y, \lambda, \nu) = \frac{1}{2}\|y\|_2^2 + \lambda\left(\|x\|_1 - 1\right) + \nu^T\left(y - x + a\right).$$

The dual function is

$$\begin{aligned} g(\lambda, \nu) &= \inf_{x,y}\left[\frac{1}{2}\|y\|_2^2 + \lambda\left(\|x\|_1 - 1\right) + \nu^T\left(y - x + a\right)\right] \\ &= \inf_x\left(\lambda\|x\|_1 - \nu^T x\right) + \inf_y\left(\frac{1}{2}y^T y + \nu^T y\right) - \lambda - a^T\nu. \end{aligned}$$

The optimization over x is

$$\inf_x\left(\lambda\|x\|_1 - \nu^T x\right) = \inf_x \sum_{i=1}^n \left(\lambda|x_i| - \nu_i x_i\right).$$

If $\lambda < \nu_i$, we will let $x_i \to \infty$, which makes the objective value unbounded below. If $\lambda \geq \nu_i$, the objective value satisfies

$$\lambda|x_i| - \nu_i x_i \geq \lambda|x_i| - \nu_i|x_i| = (\lambda - \nu_i)|x_i| \geq 0,$$

where the inequality holds with equality when $x_i = 0$. Therefore, when $\lambda \geq \nu_i$, the optimal $x_i^* = 0$, and the objective value is 0.

The optimization over y is minimization of a convex quadratic function. Therefore, the optimal $y^* = -\nu$, and the objective value is $-\frac{1}{2}\nu^T\nu$.

Combining the above two parts, the dual function is

$$g(\lambda, \nu) = \begin{cases} -\frac{1}{2}\nu^T\nu - a^T\nu - \lambda & \text{if } \nu \leq \lambda\mathbf{1} \\ -\infty & \text{otherwise} \end{cases}.$$

After making the implicit constraints explicit, the dual problem is

$$\begin{array}{ll} \text{maximize} & -\frac{1}{2}\nu^T\nu - a^T\nu - \lambda \\ \text{subject to} & \nu \leq \lambda\mathbf{1}, \\ & \lambda \geq 0, \end{array}$$

with optimization variable $\lambda \in \mathbb{R}$ and $\nu \in \mathbb{R}^n$.

5. (4 points) *Semidefinite relaxation of nonconvex problem.*
A signal $\hat{s} \in \{-1, 1\}^n$ is sent through a noisy channel, and received as $y = H\hat{s} + w$, where $H \in \mathbb{R}^{m\times n}$, $y \in \mathbb{R}^m$, and w is the Gaussian noise. The maximum likelihood detection of the signal can be formulated as the following nonconvex problem:

$$\begin{array}{ll} \text{minimize} & \|y - Hs\|_2^2 \\ \text{subject to} & s_i^2 = 1,\ i = 1, \dots, n. \end{array} \tag{2}$$

In the next series of questions, we explore the semidefinite relaxation (SDR) of the original nonconvex problem (2), and relationship between their dual problems.

(a) (1 point) Show that the original problem (2) can be rewritten in homogenous form:

$$\begin{array}{ll} \text{minimize} & x^T L x \\ \text{subject to} & x_i^2 = 1,\ i = 1, \dots, n, \\ & x_{n+1} = 1. \end{array} \tag{3}$$

Specify L as a function of the original problem data H and y. (*Hint:* define $x = \begin{bmatrix} s \\ 1 \end{bmatrix}$.)

Solution:
The objective function of the original problem (2) can be written as

$$\|y-Hs\|_2^2 = (y-Hs)^T(y-Hs) = s^T H^T H s - 2y^T H s + y^T y = \begin{bmatrix} s \\ 1 \end{bmatrix}^T \begin{bmatrix} H^T H & -H^T y \\ -y^T H & y^T y \end{bmatrix} \begin{bmatrix} s \\ 1 \end{bmatrix}.$$

Therefore, by defining $x = \begin{bmatrix} s \\ 1 \end{bmatrix}$ and $L = \begin{bmatrix} H^T H & -H^T y \\ -y^T H & y^T y \end{bmatrix}$, the original problem can be rewritten in homogenous form:

$$\begin{array}{ll} \text{minimize} & x^T L x \\ \text{subject to} & x_i^2 = 1,\ i = 1, \dots, n, \\ & x_{n+1} = 1. \end{array} \tag{4}$$

(b) (2 points) Derive the dual problem of the problem in homogenous form (5).

Solution:

The problem in (5) is equivalent to

$$\begin{array}{ll} \text{minimize} & x^T L x \\ \text{subject to} & x_i^2 = 1,\ i = 1, \dots, n+1. \end{array} \qquad (5)$$

Note that the equality constraints are all quadratic terms of x_i in this formulation, which makes it easier to derive the dual problem.
The Lagrangian is

$$L(x, \nu) = x^T L x + \sum_{i=1}^{n+1} \nu_i (x_i^2 - 1).$$

The dual function is

$$\begin{aligned} g(\nu) &= \inf_x \left[x^T L x + \sum_{i=1}^{n+1} \nu_i (x_i^2 - 1) \right] \\ &= \inf_x \left[x^T \left(L + \mathbf{diag}(\nu) \right) x - \sum_{i=1}^{n+1} \nu_i \right]. \end{aligned}$$

The dual function is unbounded below unless

$$L + \mathbf{diag}(\nu) \succeq 0.$$

Therefore, we have

$$g(\nu) = \begin{cases} -\sum_{i=1}^{n+1} \nu_i & \text{if } L + \mathbf{diag}(\nu) \succeq 0 \\ -\infty & \text{otherwise} \end{cases}.$$

The dual problem is then

$$\begin{array}{ll} \text{maximize} & -\sum_{i=1}^{n+1} \nu_i \\ \text{subject to} & L + \mathbf{diag}(\nu) \succeq 0. \end{array}$$

(c) (1 point) Now we consider the SDR of the nonconvex problem (5). We define a new variable $X = xx^T \in \mathbb{S}_+^n$. The problem in (5) is equivalent to

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(LX) \\ \text{subject to} & \mathbf{diag}(X) = \mathbf{1}_{n+1}, \\ & X \succeq 0, \\ & \mathbf{rank}(X) = 1, \end{array}$$

where $\mathbf{1}_{n+1}$ is $(n+1)$-dimensional vector of 1's. By removing the nonconvex rank constraint, we have the following SDR:

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(LX) \\ \text{subject to} & \mathbf{diag}(X) = \mathbf{1}_{n+1}, \\ & X \succeq 0, \end{array} \tag{6}$$

which is a semidefinite program (SDP).

Derive the dual problem of the SDR (6) and show that it is the same as the dual of the original problem derived in part (b). (In other words, the SDR is the dual of the dual of the original nonconvex problem.)

(*Hint:* the Lagrangian multiplier associate with the constraint $X \succeq 0$ is $Z \succeq 0$, and we need to use the term $-\mathbf{tr}(ZX)$ in the Lagrangian.)

Solution:

Write $\nu \in \mathbb{R}^{n+1}$ as the Lagrangian multiplier associated with the linear constraint $\mathbf{diag}(X) = \mathbf{1}_{n+1}$, and $Z \in \mathbb{S}_+^n$ as the Lagrangian multiplier associated with the constraint $X \succeq 0$. Then the Lagrangian is

$$L(X, Z) = \mathbf{tr}(LX) + \nu^T \left(\mathbf{diag}(X) - \mathbf{1}_{n+1}\right) - \mathbf{tr}(ZX).$$

Observing that

$$\nu^T \mathbf{diag}(X) = \mathbf{tr}\left(\mathbf{diag}(\nu) X\right),$$

the Lagrangian can be rewritten as

$$L(X, Z) = \mathbf{tr}\left(\left(L + \mathbf{diag}(\nu) - Z\right) X\right) - \mathbf{1}_{n+1}^T \nu.$$

The dual function is

$$g(Z) = \inf_X \left[\mathbf{tr}\left(\left(L + \mathbf{diag}(\nu) - Z\right) X\right) - \mathbf{1}_{n+1}^T \nu\right].$$

If $L + \mathbf{diag}(\nu) - Z \neq 0$, we can make $\mathbf{tr}\left(\left(L + \mathbf{diag}(\nu) - Z\right) X\right)$ unbounded below. If $L + \mathbf{diag}(\nu) - Z = 0$, we have $\mathbf{tr}\left(\left(L + \mathbf{diag}(\nu) - Z\right) X\right) = 0$. Therefore, the dual function is

$$g(Z) = \begin{cases} -\mathbf{1}_{n+1}^T \nu & \text{if } L + \mathbf{diag}(\nu) = Z \\ -\infty & \text{otherwise} \end{cases}.$$

The dual problem is then

$$\begin{array}{ll} \text{maximize} & -\mathbf{1}_{n+1}^T \nu \\ \text{subject to} & L + \mathbf{diag}(\nu) = Z \\ & Z \succeq 0, \end{array}$$

which is equivalent to

$$\begin{array}{ll} \text{maximize} & -\mathbf{1}_{n+1}^T \nu \\ \text{subject to} & L + \mathbf{diag}(\nu) \succeq 0, \end{array}$$

which is the same as the dual problem derived in part (b).