Solution to Homework 2

October 26, 2017

Solution to 4.7

(a)

The domain of the objective function is

$$\mathbf{dom} f_0 \cap \{x \mid c^T x + d > 0\}.$$

Since f_0 is convex, its domain $\mathbf{dom} f_0$ is convex. Therefore, the domain of the objective function is the intersection of two convex sets, and hence is convex.

The α-sublevel set of the objective function is

$$\left\{x \mid c^T x + d > 0,\ f_0(x) \leq \alpha(c^T x + d)\right\},$$

which is convex.

Therefore, the objective function is a quasiconvex function, and the problem is a quasiconvex optimization problem.

(b)

To prove the equivalence of these two problems, we need to prove that from any feasible point of one problem, we can derive a feasible point of the other problem, with the same objective value.

First, given a feasible point x of the original problem, we define

$$y = \frac{x}{c^T x + d} \text{ and } t = \frac{1}{c^T x + d} > 0.$$

Notice that

$$g_i(y, t) = t f_i(y/t) = t f_i(x),\ i = 1, \ldots, m.$$

Since t is strictly positive, we have $g_i(y, t) \leq 0$. In addition, it is not difficult to see that

$$\begin{aligned} & Ax = b \\ \Leftrightarrow\quad & A(y/t) = b \\ \Leftrightarrow\quad & Ay = bt. \end{aligned}$$

Finally, we have

$$c^T y + dt = c^T \frac{x}{c^T x + d} + d\frac{1}{c^T x + d} = \frac{c^T x + d}{c^T x + d} = 1.$$

Therefore, the variable (y, t) defined above is feasible in the reformulated problem.

For the objective value, we have

$$g_0(y, t) = tf_0(y/t) = tf_0(x) = \frac{f_0(x)}{c^T x + d}.$$

Therefore, the new variable (y, t) has the same objective value as x.

In summary, given a feasible point x of the original problem, we can find a feasible point (y, t) of the reformulated problem, with the same objective value $g_0(y, t) = f_0(x)$.

Second, given a feasible point (y, t) of the reformulated problem, we define

$$x = y/t.$$

The last equality of the reformulated problem ensures that

$$c^T x + d = c^T (y/t) + d = \frac{c^T y + dt}{t} = 1/t.$$

It is not difficult to verify that x is feasible in the original problem and has the same objective value $f_0(x) = g_0(y, t)$.

In conclusion, the two problems are equivalent.

(c)

The convex formulation is

$$\begin{array}{ll} \text{minimize} & g_0(y, t) \\ \text{subject to} & g_i(y, t) \leq 0, \ i = 1, \ldots, m \\ & Ay = bt \\ & \tilde{h}(y, t) \leq -1, \end{array}$$

where g_i is the perspective of f_i, and $\tilde{h}$ is the perspective of $-h$. Next, we verify that it is equivalent to the original problem.

First, given any feasible point x of the original problem, we define

$$y = \frac{x}{h(x)} \text{ and } t = \frac{1}{h(x)} > 0.$$

From

$$g_i(y, t) = tf_i(y/t) = tf_i(x), \ i = 1, \ldots, m,$$

and

$$
\begin{aligned}
& Ax = b \\
\Leftrightarrow \quad & A(y/t) = b \\
\Leftrightarrow \quad & Ay = bt,
\end{aligned}
$$

we can see that (y, t) is feasible for the first two constraints.

In addition, we have

$$\tilde{h}(y, t) = -th(y/t) = -th(x) = -1.$$

Therefore, the point (y, t) is feasible in the reformulated problem, with the same objective value

$$g_0(y, t) = tf_0(y/t) = f_0(x)/h(x).$$

Second, given any feasible point (y, t) of the reformulated problem, we define

$$x = y/t.$$

It is not difficult to see that x is feasible in the original problem.

For the objective value, observe that

$$\tilde{h}(y, t) = -th(y/t) = -th(x) \leq -1,$$

which implies

$$th(x) \geq 1 \text{ and } h(x) > 0.$$

Therefore, the objective value satisfies

$$\frac{f_0(x)}{h(x)} = \frac{g_0(y, t)}{th(x)} \leq g_0(y, t).$$

In summary, given any feasible point (y, t) of the reformulated problem, we can find a feasible point x of the original problem with equal or smaller objective value.

In conclusion, the two problems are equivalent.

For the example, we have

$$
\begin{aligned}
g_0(y, t) &= tf_0(y/t) \\
&= (t/m) \cdot \mathbf{tr}\left(F_0 + (y_1/t)F_1 + \cdots + (y_n/t)F_n\right) \\
&= (1/m) \cdot \mathbf{tr}\left(tF_0 + y_1F_1 + \cdots + y_nF_n\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{h}(y, t) &= -th(y/t) \\
&= -t\left[\det\left(F_0 + (y_1/t)F_1 + \cdots + (y_n/t)F_n\right)\right]^{\frac{1}{m}} \\
&= -\left[t^m \det\left(F_0 + (y_1/t)F_1 + \cdots + (y_n/t)F_n\right)\right]^{\frac{1}{m}} \\
&= -\left\{\det\left[t\left(F_0 + (y_1/t)F_1 + \cdots + (y_n/t)F_n\right)\right]\right\}^{\frac{1}{m}} \\
&= -\left[\det\left(tF_0 + y_1F_1 + \cdots + y_nF_n\right)\right]^{\frac{1}{m}}
\end{aligned}
$$

Therefore, the convex reformulation is

$$\begin{array}{ll} \text{minimize} & (1/m) \cdot \mathbf{tr}\left(tF_0 + y_1 F_1 + \cdots + y_n F_n\right) \\ \text{subject to} & \left[\det\left(tF_0 + y_1 F_1 + \cdots + y_n F_n\right)\right]^{\frac{1}{m}} \geq 1, \end{array}$$

with domain

$$\left\{(y,t) \mid t > 0,\ tF_0 + y_1 F_1 + \cdots + y_n F_n \succ 0\right\}.$$

Solution to 4.11

We write a_i^T as the ith row of the matrix A, and b_i as the ith element of the vector b.

(a)

The problem is equivalent to

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \max\limits_{i=1,\ldots,m} \left|a_i^T x - b_i\right| \leq t,\ i = 1, \ldots, m, \end{array}$$

which is equivalent to the following LP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & a_i^T x - b_i \leq t,\ i = 1, \ldots, m \\ & a_i^T x - b_i \geq -t,\ i = 1, \ldots, m, \end{array}$$

which can be rewritten compactly as

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax - b \leq t\mathbf{1}_m, \\ & Ax - b \geq -t\mathbf{1}_m, \end{array}$$

with optimization variables $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

(b)

The problem is equivalent to

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{m} t_i \\ \text{subject to} & \left|a_i^T x - b_i\right| \leq t_i,\ i = 1, \ldots, m, \end{array}$$

which is equivalent to the following LP

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T t \\ \text{subject to} & Ax - b \leq t, \\ & Ax - b \geq -t, \end{array}$$

with optimization variables $t \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$.

(c)

The problem is equivalent to

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{m} t_i \\ \text{subject to} & \left|a_i^T x - b_i\right| \le t_i,\ i = 1, \ldots, m, \\ & \displaystyle\max_{i=1,\ldots,n} |x_i| \le 1, \end{array}$$

which is equivalent to the following LP

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T t \\ \text{subject to} & Ax - b \le t, \\ & Ax - b \ge -t, \\ & x \le \mathbf{1}, \\ & x \ge -\mathbf{1}, \end{array}$$

with optimization variables $t \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$.

(d)

The problem is equivalent to

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{m} t_i \\ \text{subject to} & |x_i| \le t_i,\ i = 1, \ldots, m, \\ & \displaystyle\max_{i=1,\ldots,n} \left|a_i^T x - b_i\right| \le 1, \end{array}$$

which is equivalent to the following LP

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T t \\ \text{subject to} & x \le t, \\ & x \ge -t, \\ & Ax - b \le \mathbf{1}, \\ & Ax - b \ge -\mathbf{1}, \end{array}$$

with optimization variables $t \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$.

(e)

The problem is equivalent to

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{m} t_i + s \\ \text{subject to} & \left|a_i^T x - b_i\right| \le t_i,\ i = 1, \ldots, m, \\ & \displaystyle\max_{i=1,\ldots,n} |x_i| \le s, \end{array}$$

which is equivalent to the following LP

$$
\begin{array}{ll}
\text{minimize} & \mathbf{1}^T t + s \\
\text{subject to} & Ax - b \preceq t, \\
& Ax - b \succeq -t, \\
& x \preceq s\mathbf{1}, \\
& x \succeq -s\mathbf{1},
\end{array}
$$

with optimization variables $t \in \mathbb{R}^m$, $s \in \mathbb{R}$, and $x \in \mathbb{R}^n$.

Solution to 4.15

(a)

The feasible set of the LP relaxation includes the feasible set of the Boolean LP. Therefore, the optimal value of the LP relaxation is a lower bound on the optimal value of the Boolean LP. Moreover, if the LP relaxation is infeasible, the Boolean LP is infeasible.

(b)

In this case, the optimal solution of the LP relaxation is also the optimal solution of the Boolean LP, or the LP relaxation is exact.

Solution to 4.23

The problem is equivalent to

$$
\text{minimize} \quad \sum_{i=1}^{m} \left(a_i^T x - b_i\right)^4,
$$

which is equivalent to

$$
\begin{array}{ll}
\text{minimize} & \displaystyle\sum_{i=1}^{m} t_i^2 \\
\text{subject to} & \left(a_i^T x - b_i\right)^2 \le t_i, \ i = 1, \ldots, m,
\end{array}
$$

which is equivalent to the QCQP

$$
\begin{array}{ll}
\text{minimize} & \displaystyle\sum_{i=1}^{m} t_i^2 \\
\text{subject to} & x_i^T \left(a_i a_i^T\right) x - 2(b_i a_i)^T x + b_i^2 \le t_i, \ i = 1, \ldots, m,
\end{array}
$$

with optimization variables $t \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$.

Solution to 4.33

(a)

The problem is equivalent to

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & p(x) \leq t, \\ & q(x) \leq t, \end{array}$$

which is equivalent to the following GP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & p(x)/t \leq 1, \\ & q(x)/t \leq 1. \end{array}$$

(b)

The problem is equivalent to

$$\begin{array}{ll} \text{minimize} & \exp(t_1) + \exp(t_2) \\ \text{subject to} & p(x) \leq t_1, \\ & q(x) \leq t_2. \end{array}$$

With the change of variables $y_i = \log x_i$, the constraints can be made convex. The objective function is convex in (t_1, t_2). The resulting problem will be convex.

(c)

The problem is equivalent to

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \dfrac{p(x)}{r(x) - q(x)} \leq t, \\ & r(x) > q(x), \end{array}$$

which is equivalent to

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & p(x) \leq t\left(r(x) - q(x)\right), \\ & q(x) \leq r(x). \end{array}$$

which is equivalent to

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & p(x) + tq(x) \leq tr(x), \\ & q(x) \leq r(x). \end{array}$$

which is equivalent to the following GP

$$\begin{array}{ll}\text{minimize} & t \\ \text{subject to} & p(x)/(t \cdot r(x)) + q(x)/r(x) \leq 1, \\ & q(x)/r(x) \leq 1.\end{array}$$

Solution to 5.2

First, consider the case when $p^\star = -\infty$.

The Lagrangian is

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x).$$

Since the optimal value $p^\star = -\infty$, we can find feasible points x that make the objective value $f_0(x)$ arbitrarily small. Therefore, for any $\lambda \geq 0$ and ν, the Lagrangian $L(x, \lambda, \nu)$ is unbounded below. Hence, the dual function is

$$g(\lambda, \nu) = \inf L(x, \lambda, \nu) = -\infty.$$

The optimal value of the dual problem is $d^\star = -\infty$. In other words, the weak duality holds.

Next, consider the case when $d^\star = \infty$.

Suppose that there was a feasible point $\tilde{x}$ of the primal problem. Then for any $\lambda \geq 0$ and ν, we have

$$g(\lambda, \nu) = \inf \left[f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right] \leq f_0(\tilde{x}) + \sum_{i=1}^{m} \lambda_i f_i(\tilde{x}).$$

This is contradictory to the fact that $d^\star = \infty$. Therefore, the primal problem is infeasible. As a result, we have $p^\star = \infty$. In other words, the weak duality holds.

Solution to 5.5

The Lagrangian is

$$L(x, \lambda, \nu) = c^T x + \lambda^T (Gx - h) + \nu^T (Ax - b).$$

The dual function is

$$\begin{aligned} g(\lambda, \nu) &= \inf L(x, \lambda, \nu) \\ &= \inf \left[c^T x + \lambda^T (Gx - h) + \nu^T (Ax - b) \right] \\ &= \inf \left[\left(c + G^T \lambda + A^T \nu \right)^T x - h^T \lambda - b^T \nu \right] \\ &= \begin{cases} -h^T \lambda - b^T \nu & \text{if } c + G^T \lambda + A^T \nu = 0 \\ -\infty & \text{otherwise} \end{cases}. \end{aligned}$$

The dual problem is

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \geq 0. \end{array}$$

After making the implicit constraints explicit, we have

$$\begin{array}{ll} \text{maximize} & -h^T\lambda - b^T\nu \\ \text{subject to} & \lambda \geq 0 \\ & c + G^T\lambda + A^T\nu = 0. \end{array}$$

Solution to 5.11

The problem after introducing the new variables $y_i = A_i x + b_i$ is

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{N} \|y_i\|_2 + (1/2)\|x - x_0\|_2^2 \\ \text{subject to} & y_i = A_i x + b_i, \; i = 1, \ldots, N. \end{array}$$

The Lagrangian is

$$L(x, y, \nu) = \sum_{i=1}^{N} \|y_i\|_2 + (1/2)\|x - x_0\|_2^2 + \sum_{i=1}^{N} \nu_i^T (y_i - A_i x - b_i).$$

The dual function is

$$\begin{aligned} g(\nu) &= \inf_{x,y} L(x, y, \nu) \\ &= \inf_{x,y} \left[\sum_{i=1}^{N} \|y_i\|_2 + (1/2)\|x - x_0\|_2^2 + \sum_{i=1}^{N} \nu_i^T (y_i - A_i x - b_i) \right] \\ &= \inf_{x} \left[(1/2)\|x - x_0\|_2^2 - \left(\sum_{i=1}^{N} A_i^T \nu_i \right)^T x \right] + \inf_{y} \left(\sum_{i=1}^{N} \|y_i\|_2 + \sum_{i=1}^{N} \nu_i^T y_i \right) - \sum_{i=1}^{N} b_i^T \nu_i. \end{aligned}$$

For the infimum related to x, we have

$$\begin{aligned} & \inf_{x} \left[(1/2)\|x - x_0\|_2^2 - \left(\sum_{i=1}^{N} A_i^T \nu_i \right)^T x \right] \\ = & \inf_{x} \left[(1/2)x^T x - \left(\sum_{i=1}^{N} A_i^T \nu_i + x_0 \right)^T x + (1/2)x_0^T x_0 \right]. \end{aligned}$$

Since the objective function is convex, the first-order derivative is zero at the optimal solution. By taking the derivative and setting it to zero, we get the optimal solution

$$x^*(\nu) = x_0 + \sum_{i=1}^{N} A_i^T \nu_i,$$

and the optimal value

$$\begin{aligned}
& -(1/2)\left(x_0 + \sum_{i=1}^{N} A_i^T \nu_i\right)^T \left(x_0 + \sum_{i=1}^{N} A_i^T \nu_i\right) + (1/2) x_0^T x_0 \\
= \quad & -\sum_{i=1}^{N} \left(A_i^T \nu_i\right)^T x_0 - (1/2)\left(\sum_{i=1}^{N} A_i^T \nu_i\right)^T \left(\sum_{i=1}^{N} A_i^T \nu_i\right) \\
= \quad & -\textstyle\sum_{i=1}^{N} \left(A_i x_0\right)^T \nu_i - (1/2)\left\|\sum_{i=1}^{N} A_i^T \nu_i\right\|_2^2 .
\end{aligned}$$

For the infimum related to y, we have

$$\begin{aligned}
& \inf_y \left(\sum_{i=1}^{N} \|y_i\|_2 + \sum_{i=1}^{N} \nu_i^T y_i\right) \\
= \quad & \sum_{i=1}^{N} \inf_{y_i} \left(\|y_i\|_2 + \nu_i^T y_i\right) .
\end{aligned}$$

The optimal $y_i^*(\nu)$ is determined by its norm and its direction. Fixing the norm of y_i, we let y_i have the direction of $-\nu_i$ to minimize $\nu_i^T y_i$. Therefore, the optimal $y_i^*(\nu)$ should have the form

$$y_i^*(\nu) = -\alpha_i \nu_i,$$

where $\alpha_i \geq 0$ is a scalar. Then the optimization problem of y_i reduces to an optimization problem of α_i

$$\begin{aligned}
& \inf_{y_i} \left(\|y_i\|_2 + \nu_i^T y_i\right) \\
= \quad & \inf_{\alpha_i} \left(\alpha_i \|\nu_i\|_2 - \alpha_i \nu_i^T \nu_i\right) \\
= \quad & \inf_{\alpha_i} \left(\alpha_i \|\nu_i\|_2 - \alpha_i \|\nu_i\|_2^2\right) .
\end{aligned}$$

If $\|\nu_i\|_2 > 1$, we can set α_i arbitrarily large to make the optimal value $-\infty$. If $\|\nu_i\|_2 \leq 1$, we can set $\alpha_i = 0$ to minimize the objective. Therefore, we have

$$\inf_{y_i} \left(\|y_i\|_2 + \nu_i^T y_i\right) = \begin{cases} 0 & \text{if } \|\nu_i\|_2 \leq 1 \\ -\infty & \text{otherwise} \end{cases} .$$

In conclusion, the dual function is

$$\begin{aligned}
g(\nu) \quad = \quad & \begin{cases} -\sum_{i=1}^{N} \left(A_i x_0\right)^T \nu_i - (1/2)\left\|\sum_{i=1}^{N} A_i^T \nu_i\right\|_2^2 - \sum_{i=1}^{N} b_i^T \nu_i & \text{if } \|\nu_i\|_2 \leq 1 \\ -\infty & \text{otherwise} \end{cases} \\
= \quad & \begin{cases} -\sum_{i=1}^{N} \left(A_i x_0 + b_i\right)^T \nu_i - (1/2)\left\|\sum_{i=1}^{N} A_i^T \nu_i\right\|_2^2 & \text{if } \|\nu_i\|_2 \leq 1 \\ -\infty & \text{otherwise} \end{cases} .
\end{aligned}$$

Therefore, the dual problem is

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^N \left(A_i x_0 + b_i\right)^T \nu_i - (1/2) \left\| \sum_{i=1}^N A_i^T \nu_i \right\|_2^2 \\ \text{subject to} & \|\nu_i\|_2 \leq 1,\ i = 1, \ldots, N. \end{array}$$

Solution to 5.21

(a)

The objective function is convex. The constraint is the perspective of function z^2, which is convex. Therefore, the problem is a convex optimization problem.

Since any feasible point (x, y) must satisfy $x = 0$, the optimal value is $p^\star = 1$.

(b)

The Lagrangian is

$$L(x, \lambda) = e^{-x} + \lambda \frac{x^2}{y}.$$

The dual function is

$$\begin{aligned} g(\lambda) &= \inf L(x, \lambda) \\ &= \inf \left(e^{-x} + \lambda \frac{x^2}{y} \right). \end{aligned}$$

For any $\lambda \geq 0$, we can set $y = x^3$ and let $x \to \infty$. Then we can get

$$\lim_{x \to \infty} \left(e^{-x} + \lambda \frac{x^2}{y} \right) = \lim_{x \to \infty} \left(e^{-x} + \lambda \frac{1}{x} \right) = 0.$$

Therefore, the dual function is constant, namely $g(\lambda) = 0$ for all $\lambda \geq 0$. The optimal value of the dual problem is $d^\star = 0$. The duality gap is 1.

(c)

Since the feasible set is a line $\{(x, y) \mid x = 0\}$ and has no interior, there is no strictly feasible point. Slater's condition does not hold.

Solution to 5.26

(a)

The first constraint depicts a unit ball centered at $(1, 1)$. The second constraint depicts a unit ball centered at $(1, -1)$. The intersection of the two unit balls is one point $(1, 0)$. Therefore, there is only one feasible point $(1, 0)$. The optimal solution is $x^\star = (1, 0)$, and the optimal value is $p^\star = 1$.

(b)

The Lagrangian is

$$L(x_1, x_2, \lambda_1, \lambda_2) = x_1^2 + x_2^2 + \lambda_1 \left[(x_1 - 1)^2 + (x_2 - 1)^2 - 1\right] + \lambda_2 \left[(x_1 - 1)^2 + (x_2 + 1)^2 - 1\right].$$

The derivatives of the Lagrangian are

$$\frac{\partial L(x_1, x_2, \lambda_1, \lambda_2)}{\partial x_1} = 2x_1 + 2\lambda_1(x_1 - 1) + 2\lambda_2(x_1 - 1)$$

and

$$\frac{\partial L(x_1, x_2, \lambda_1, \lambda_2)}{\partial x_2} = 2x_2 + 2\lambda_1(x_2 - 1) + 2\lambda_2(x_2 + 1)$$

The KKT conditions are

$$\begin{aligned}
2x_1 + 2\lambda_1(x_1 - 1) + 2\lambda_2(x_1 - 1) &= 0, \\
2x_2 + 2\lambda_1(x_2 - 1) + 2\lambda_2(x_2 + 1) &= 0, \\
(x_1 - 1)^2 + (x_2 - 1)^2 &\leq 1, \\
(x_1 - 1)^2 + (x_2 + 1)^2 &\leq 1, \\
\lambda_1 &\geq 0, \\
\lambda_2 &\geq 0, \\
\lambda_1 \left[(x_1 - 1)^2 + (x_2 - 1)^2 - 1\right] &= 0, \\
\lambda_2 \left[(x_1 - 1)^2 + (x_2 + 1)^2 - 1\right] &= 0.
\end{aligned}$$

At the optimal solution $x^\star = (1, 0)$, the KKT conditions reduce to

$$\begin{aligned}
2 &= 0, \\
-2\lambda_1 + 2\lambda_2 &= 0, \\
\lambda_1 &\geq 0, \\
\lambda_2 &\geq 0,
\end{aligned}$$

which has no solution.

Therefore, there is no Lagrange multipliers (λ_1, λ_2) that prove that $x^\star$ is optimal.

(b)

The Lagrangian is

$$L(x_1, x_2, \lambda_1, \lambda_2) = x_1^2 + x_2^2 + \lambda_1 \left[(x_1 - 1)^2 + (x_2 - 1)^2 - 1\right] + \lambda_2 \left[(x_1 - 1)^2 + (x_2 + 1)^2 - 1\right].$$

The dual function is

$$
\begin{aligned}
g(\lambda_1, \lambda_2) &= \inf L(x_1, x_2, \lambda_1, \lambda_2) \\
&= \inf_{x_1} \left[x_1^2 + \lambda_1(x_1 - 1)^2 + \lambda_2(x_1 - 1)^2\right] + \inf_{x_2} \left[x_2^2 + \lambda_1(x_2 - 1)^2 + \lambda_2(x_2 + 1)^2\right] + \lambda_1 + \lambda_2 \\
&= \inf_{x_1} \left[(1 + \lambda_1 + \lambda_2)x_1^2 - 2(\lambda_1 + \lambda_2)x_1 + \lambda_1 + \lambda_2\right] \\
&+ \inf_{x_2} \left[(1 + \lambda_1 + \lambda_2)x_2^2 - 2(\lambda_1 - \lambda_2)x_2 + \lambda_1 + \lambda_2\right] \\
&+ \lambda_1 + \lambda_2.
\end{aligned}
$$

If $1 + \lambda_1 + \lambda_2 \le 0$, the objective functions are linear or concave, and hence the dual function is unbounded below. If $1 + \lambda_1 + \lambda_2 > 0$, the objective functions are convex, and the optimal solutions are

$$x_1^*(\lambda_1, \lambda_2) = \frac{\lambda_1 + \lambda_2}{1 + \lambda_1 + \lambda_2},$$

and

$$x_2^*(\lambda_1, \lambda_2) = \frac{\lambda_1 - \lambda_2}{1 + \lambda_1 + \lambda_2}.$$

Plugging in the expressions of $x_1^*(\lambda_1, \lambda_2)$ and $x_2^*(\lambda_1, \lambda_2)$, we have

$$g(\lambda_1, \lambda_2) = \begin{cases} -\frac{(\lambda_1+\lambda_2)^2}{1+\lambda_1+\lambda_2} - \frac{(\lambda_1-\lambda_2)^2}{1+\lambda_1+\lambda_2} + \lambda_1 + \lambda_2 & \text{if } 1 + \lambda_1 + \lambda_2 > 0 \\ -\infty & \text{otherwise} \end{cases}.$$

The dual problem is

$$
\begin{array}{ll}
\text{maximize} & -\dfrac{(\lambda_1 + \lambda_2)^2}{1 + \lambda_1 + \lambda_2} - \dfrac{(\lambda_1 - \lambda_2)^2}{1 + \lambda_1 + \lambda_2} + \lambda_1 + \lambda_2 \\
\text{subject to} & 1 + \lambda_1 + \lambda_2 > 0.
\end{array}
$$

We define $z = \lambda_1 + \lambda_2$. Then the dual problem is equivalent to

$$
\begin{array}{ll}
\text{maximize} & -\dfrac{z^2}{1 + z} - \dfrac{(\lambda_1 - \lambda_2)^2}{1 + z} + z \\
\text{subject to} & 1 + z > 0 \\
& \lambda_1 + \lambda_2 = z.
\end{array}
$$

From the second term of the objective function, we can see that the optimal $(\lambda_1^\star, \lambda_2^\star)$ must satisfy $\lambda_1^\star = \lambda_2^\star$. Therefore, we can set $\lambda_1 = \lambda_2 = \lambda$ and $z = 2\lambda$. The dual problem reduces to

$$
\begin{array}{ll}
\text{maximize} & -4\dfrac{\lambda^2}{1 + 2\lambda} + 2\lambda \\
\text{subject to} & 1 + 2\lambda > 0,
\end{array}
$$

which is equivalent to

$$
\begin{array}{ll}
\text{maximize} & \dfrac{2\lambda}{1+2\lambda} \\
\text{subject to} & 1+2\lambda > 0
\end{array}
$$

By letting $\lambda \to \infty$, the objective value of the dual problem approaches 1. Therefore, we have $d^\star = 1 = p^\star$. However, the dual optimum is not attained.

In this case, strong duality holds. (However, since the dual optimum is not attained, the KKT conditions do not hold.)