Solution to Homework 1

October 25, 2017

Solution to 2.7

We rewrite the inequality as follows:

$$\begin{aligned}
& \|x-a\|_2 \leq \|x-b\|_2 \\
\Leftrightarrow\ & \|x-a\|_2^2 \leq \|x-b\|_2^2 \\
\Leftrightarrow\ & (x-a)^T(x-a) \leq (x-b)^T(x-b) \\
\Leftrightarrow\ & x^Tx - 2a^Tx + a^Ta \leq x^Tx - 2b^Tx + b^Tb \\
\Leftrightarrow\ & 2(b-a)^Tx \leq b^Tb - a^Ta.
\end{aligned}$$

Therefore, the set can be rewritten as

$$\left\{x \mid 2(b-a)^Tx \leq b^Tb - a^Ta\right\},$$

which is a halfspace.

Solution to 2.11

We show this by definition for the general case in $\mathbb{R}^n$.

Suppose that $x, y \in \mathbb{R}^n$ are in the n-dimensional hyperbolic set. We look at $\theta x + (1-\theta)y$ for $\theta \in [0,1]$.

$$\begin{aligned}
\prod_{i=1}^n [\theta x_i + (1-\theta)y_i] &\geq \prod_{i=1}^n \left(x_i^\theta y_i^{1-\theta}\right) \\
&= \prod_{i=1}^n x_i^\theta \cdot \prod_{i=1}^n y_i^{1-\theta} \\
&= \left(\prod_{i=1}^n x_i\right)^\theta \cdot \left(\prod_{i=1}^n y_i\right)^{1-\theta} \\
&\geq 1,
\end{aligned}$$

where the first equality follows from the hint.

Therefore, $\theta x + (1-\theta)y$ is in the n-dimensional hyperbolic set. This concludes the proof that the n-dimensional hyperbolic set is convex.

Solution to 2.12

(a)

A slab is the intersection of two halfspaces, and therefore is convex.

(b)

A hyperrectangle in n-dimensional space is the intersection of $2n$ halfspaces, and therefore is convex.

(c)

A wedge is the intersection of two halfspaces, and therefore is convex.

(d)

We rewrite the set as follows:

$$\begin{aligned} & \{x \mid \|x - x_0\|_2 \le \|x - y\|_2 \text{ for all } y \in S\} \\ = \quad & \bigcap_{y \in S} \{x \mid \|x - x_0\|_2 \le \|x - y\|_2\}. \end{aligned}$$

We have shown in 2.7 that the set

$$\{x \mid \|x - x_0\|_2 \le \|x - y\|_2\}$$

is a halfspace.

Therefore, the set of interest is the intersection of infinitely many halfspaces, and is convex.

(e)

This set is not convex.

A counter example is as follows. Consider the real numbers (i.e., $n = 1$). Suppose that $S = \{-1, 1\}$, and $T = \{0\}$. In other words, S contains two points in the real axis, and T contains one point. The set of interest is then $(-\infty, -0.5] \cup [0.5, \infty)$, which is not convex.

(g)

We rewrite the inequality as follows:

$$\begin{aligned} & \|x - a\|_2 \le \theta \|x - b\|_2 \\ \Leftrightarrow \quad & \|x - a\|_2^2 \le \theta^2 \|x - b\|_2^2 \\ \Leftrightarrow \quad & (x - a)^T (x - a) \le \theta^2 (x - b)^T (x - b) \\ \Leftrightarrow \quad & x^T x - 2a^T x + a^T a \le \theta^2 \left(x^T x - 2b^T x + b^T b\right) \\ \Leftrightarrow \quad & \left(1 - \theta^2\right) x^T x + 2\left(\theta^2 b - a\right)^T x + \left(a^T a - \theta^2 b^T b\right) \le 0. \end{aligned}$$

If $\theta = 1$, the set of interest is a halfspace. If $\theta \in [0, 1)$, the set of interest is a ball. Therefore, the set is convex.

Solution to 2.19

(a)

We have

$$\begin{aligned} f^{-1}(C) &= \left\{ x \in \mathbf{dom} f \mid g^T \frac{Ax+b}{c^T x + d} \leq h \right\} \\ &= \left\{ x \mid c^T x + d > 0,\ g^T(Ax+b) \leq h(c^T x + d) \right\} \\ &= \left\{ x \mid c^T x + d > 0,\ (A^T g - hc)^T x \leq hd - g^T b \right\}. \end{aligned}$$

Therefore, the set is the intersection of two halfspaces.

(b)

We have

$$\begin{aligned} f^{-1}(C) &= \left\{ x \in \mathbf{dom} f \mid G \frac{Ax+b}{c^T x + d} \leq h \right\} \\ &= \left\{ x \mid c^T x + d > 0,\ G(Ax+b) \leq h(c^T x + d) \right\} \\ &= \left\{ x \mid c^T x + d > 0,\ (GA - hc^T)x \leq dh - Gb \right\}. \end{aligned}$$

Therefore, the set is the intersection of a halfspace and a polyhedron.

(c)

We have

$$\begin{aligned} f^{-1}(C) &= \left\{ x \in \mathbf{dom} f \mid \left(\frac{Ax+b}{c^T x + d} \right)^T P^{-1} \frac{Ax+b}{c^T x + d} \leq 1 \right\} \\ &= \left\{ x \mid c^T x + d > 0,\ (Ax+b)^T P^{-1}(Ax+b) \leq (c^T x + d)^2 \right\} \\ &= \left\{ x \mid c^T x + d > 0,\ x^T Q x + q^T x + r \leq 0 \right\}, \end{aligned}$$

where $Q = A^T P^{-1} A - cc^T$, $q = 2\left(b^T P^{-1} A + dc\right)$, and $r = b^T P^{-1} b - d^2$. Therefore, when $Q \succeq 0$, the set is the intersection of a halfspace and a ellipsoid.

(d)

We write the ith row of A as a_i^T and the ith element of b as b_i. Then we have

$$y_i = a_i^T x + b_i.$$

As a result, the inverse image is

$$
\begin{aligned}
f^{-1}(C) &= \left\{ x \in \mathbf{dom} f \mid \frac{a_1^T x + b_1}{c^T x + d} A_1 + \cdots + \frac{a_n^T x + b_n}{c^T x + d} A_n \preceq B \right\} \\
&= \left\{ x \mid c^T x + d > 0,\ (a_1^T x + b_1) A_1 + \cdots + (a_n^T x + b_n) A_n \preceq B(c^T x + d) \right\}.
\end{aligned}
$$

Realizing that

$$
a_i^T x A_i = \left(\sum_{j=1}^{m} a_{i,m} x_m \right) A_i \ \text{ and } \ B c^T x = \left(\sum_{j=1}^{m} c_m x_m \right) B,
$$

we have

$$
\begin{aligned}
& \left\{ x \mid c^T x + d > 0,\ (a_1^T x + b_1) A_1 + \cdots + (a_n^T x + b_n) A_n \preceq B(c^T x + d) \right\} \\
= & \left\{ x \mid c^T x + d > 0,\ x_1 G_1 + \cdots + x_n G_n \succeq H \right\},
\end{aligned}
$$

where

$$
G_i = \sum_{j=1}^{n} a_{j,i} A_j - c_i B \ \text{ and } \ H = dB - \sum_{j=1}^{n} b_j A_j.
$$

Therefore, the set is the intersection of a halfspace and the solution set to a LMI.

Solution to 2.24

(a)

The convex set is the intersection of halfspaces, each of which is defined by the supporting hyperplane.

Given any point $(\hat{x}_1, \hat{x}_2)$ on the bound of the set, we derive the expression of the supporting hyperplane at this point. We can think of the boundary as a function $x_2 = f(x_1) = \frac{1}{x_1}$. Then the supporting hyperplane at $(\hat{x}_1, \hat{x}_2)$ is

$$
x_2 = \hat{x}_2 + f'(\hat{x}_1)(x_1 - \hat{x}_1),
$$

where $f'(\hat{x}_1)$ is the first-order derivative at $\hat{x}_1$, and can be calculated as

$$
f'(\hat{x}_1) = -\frac{1}{\hat{x}_1^2}.
$$

In addition, since $(\hat{x}_1, \hat{x}_2)$ is on the bound of the set, we have $\hat{x}_2 = \frac{1}{\hat{x}_1}$. Therefore, the supporting hyperplane is

$$
x_2 = \frac{1}{\hat{x}_1} - \frac{1}{\hat{x}_1^2}(x_1 - \hat{x}_1) = \frac{2}{\hat{x}_1} - \frac{x_1}{\hat{x}_1^2}.
$$

The halfspace that contains the convex set is then

$$\left\{(x_1, x_2) \mid x_2 \geq \frac{2}{\hat{x}_1} - \frac{x_1}{\hat{x}_1^2}\right\}.$$

Finally, we can write the convex set as

$$\bigcup_{\hat{x}_1 \in (0,\infty)} \left\{(x_1, x_2) \mid x_2 \geq \frac{2}{\hat{x}_1} - \frac{x_1}{\hat{x}_1^2}\right\}.$$

(b)

Since $\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$, the ℓ_∞-norm unit ball is a polyhedron. The boundary of the polyhedron is the set of all the points x_0 satisfying

$$\exists\ i \text{ such that } |x_{0,i}| = 1.$$

By definition, the supporting hyperplane at x_0 is

$$\left\{x \mid a^T x = a^T x_0\right\},$$

where a satisfies

$$a^T x \geq a^T x_0$$

for all x in the ℓ_∞-norm unit ball. We need to find a for every boundary point x_0.

Suppose that the boundary point satisfies

$$x_{0,i} \begin{cases} = 1 & i \in \mathcal{I}_1 \subseteq \{1, \ldots, n\} \\ = -1 & i \in \mathcal{I}_2 \subseteq \{1, \ldots, n\} \\ \in (-1, 1) & i \notin \mathcal{I}_1 \cup \mathcal{I}_2 \end{cases}.$$

Then a defines a hyperplane at x_0 if and only if

$$a_i \begin{cases} \leq 0 & i \in \mathcal{I}_1 \subseteq \{1, \ldots, n\} \\ \geq 0 & i \in \mathcal{I}_2 \subseteq \{1, \ldots, n\} \\ = 0 & i \notin \mathcal{I}_1 \cup \mathcal{I}_2 \end{cases}.$$

Solution to 3.2

For any convex function, we must have

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\left(f(x) + f(y)\right).$$

If x is in 1-level set and y is in 3-level set, the midpoint on the line between x and y must satisfy

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\left(f(x) + f(y)\right) = 2.$$

In other words, the midpoint is closer to 1-level set than to 3-level set.

Similarly, for a concave function, the midpoint on the line between a point in 1-level set and a point in 3-level set should be closer to 3-level set than to 1-level set.

For the first figure, if we pick the points on the east sides of the 1-level set and the 3-level set, the midpoint is closer to the 3-level set. Hence, the function is not convex. Similarly, if we pick the points on the southwest corners of the 1-level set and the 3-level set, the midpoint is closer to the 1-level set. Hence, the function is not concave.

The sublevel sets of the first figure are convex. By definition, the function can be quasiconvex.

The sublevel set of function f is the complement set of the sublevel set of function $-f$. Since the complement of the sublevel sets are nonconvex, the function is not quasiconcave.

We use the above guidelines to check the convexity of the function depicted by the second figure. When $\alpha < \beta$, the midpoints between a point in α-level set and a point in β-level set seem to be closer to β-level set. Therefore, the function can be concave and hence can be quasiconcave. The midpoints between a point in 3-level set and a point in 5-level set seem to be closer to 5-level set. Therefore, the function is not convex. The sublevel sets are nonconvex. Therefore, the function is not quasiconvex.

In summary, the function in the first figure can only be quasiconvex, while the function in the second figure can be concave and quasiconcave.

Solution to 3.19

(a)

We note that

$$f(x) = \sum_{i=1}^{r} \alpha_i x_{[i]} = \sum_{k=1}^{r-1} (\alpha_k - \alpha_{k+1}) \sum_{i=1}^{k} x_{[i]} + \alpha_r \sum_{i=1}^{r} x_{[i]}.$$

Since $\alpha_k \geq \alpha_{k+1}$, the function $f(x)$ is a nonnegative weighted sum of convex functions, and therefore is convex.

(b)

Since $T(x, \omega)$ is affine in x, the logarithm $\log T(x, \omega)$ is convex in x. Hence, $-\log T(x, \omega)$ is convex in x. Since the function $f(x)$ is integral of convex functions, it is convex.

Solution to 3.20

(a)

The function $f(x)$ is the norm of an affine function. Hence, $f(x)$ is convex.

(b)

The function $f(x)$ is composition of $-\det(y)^{1/n}$ and affine function $y = A_0 + x_1A_1 + \cdots x_nA_n$. Since $-\det(y)^{1/n}$ is convex (see problem 3.8), the function $f(x)$ is convex.

(c)

The function $f(X)$ is composition of $-\mathrm{tr}\left(Y^{-1}\right)$ and affine function $Y = A_0 + x_1A_1 + \cdots x_nA_n$. Since $-\mathrm{tr}\left(Y^{-1}\right)$ is convex, the function $f(x)$ is convex.

Solution to 3.21

(a)

The norm of an affine function is convex. Therefore, the function $f(x)$ is the pointwise maximum of convex functions, and hence is convex.

(b)

The pointwise absolute value $|x|$ is convex. Therefore, the function $f(x)$ is the pointwise maximum of convex functions, and hence is convex.

Solution to 3.23

(a)

The function $f(x,t)$ is the perspective of function $g(y) = \|y\|_p^p$. Since norm $\|y\|_p$ is convex, and z^p is increasing and convex in z, we have that $\|y\|_p^p$ is convex. Therefore, its perspective function $f(x,t)$ is convex.

(b)

The square of the norm function $\|z\|_2^2$ is convex. Hence, its perspective $\frac{\|y\|_2^2}{t}$ is convex. The function $f(x)$ is composition of $\frac{\|y\|_2^2}{t}$ and affine functions $y = Ax+b$ and $t = c^Tx + d$. Therefore, the function $f(x)$ is convex.