EE 617
Fall 2017
Mid-Term Exam
Oct 25, 2017
Time Limit: 120 minutes

Name (Print): ______________________________

This exam contains 5 problems. Print your name on the cover page, and put your initials on the top of every page, in case the pages become separated.

You may use your books, notes, or anything that is not electronic on this exam.

Please support your answers with calculations and explanation. A *correct answer without explanation* will receive *no* credit.

Problem	Points	Score
1	4	
2	4	
3	4	
4	4	
5	4	
Total:	20	

1. (4 points) *Convex sets.*

   (a) (2 points) Consider the set

   $$\left\{a \in \mathbb{R}^k \mid p(0) = 1,\ |p(t)| \le 1 \text{ for } \alpha \le t \le \beta\right\},$$

   where $p(t) = a_1 + a_2 t + \cdots + a_k t^{k-1}$. Is this set convex?

   (b) (2 points) The polar of a set $C \subseteq \mathbb{R}^n$ is defined as

   $$C^\circ = \left\{y \in \mathbb{R}^n \mid y^T x \le 1 \text{ for all } x \in C\right\}.$$

   Is the polar C° of a (possibly nonconvex) set C convex?

2. (4 points) *Convex functions.*

   (a) (2 points) The logarithmic barrier for the second-order cone constraint $\|x\|_2 \leq t$ is

   $$f(x,t) = -\log\left(t^2 - x^T x\right), \text{ with } \mathbf{dom} f = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\|_2 < t\}.$$

   Show that the function $f(x,t)$ is convex in (x,t). (*Hint:* use the convexity of $x^T x/t$.)

   (b) (2 points) The directional derivative of a function $g : \mathbb{R}^n \to \mathbb{R}$ is

   $$f(x) = \inf_{\alpha > 0} \frac{g(y + \alpha x) - g(y)}{\alpha}.$$

   Show that $f : \mathbb{R}^n \to \mathbb{R}$ is convex if g is convex.

3. (4 points) *Convex optimization problems.*

   (a) (2 points) Consider the following optimization problem:

$$\begin{array}{ll} \text{minimize} & \dfrac{\max_{i=1,\ldots,m}\left(a_i^T x + b_i\right)}{\min_{i=1,\ldots,p}\left(c_i^T x + d_i\right)} \\ \text{subject to} & Fx \leq g, \end{array}$$

   with variables $x \in \mathbb{R}^n$. Assume that $c_i^T x + d_i > 0$ for all x that satisfy $Fx \leq g$. Give the best convex reformulation of the problem. By "best", we mean LP is better than SOCP, SOCP is better than SDP, and SDP is better than general convex program.

(b) (2 points) Consider the following complex least ℓ_∞-norm problem:

$$\begin{array}{ll} \text{minimize} & \|x\|_\infty \\ \text{subject to} & Ax = b, \end{array}$$

with *complex* variables $x \in \mathbb{C}^n$ and *complex* parameters $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$. Note that

$$\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|.$$

Formulate the complex least ℓ_∞-norm problem as a problem with real parameters and real variables. Moreover, give the best convex reformulation of the new problem. By "best", we mean LP is better than SOCP, SOCP is better than SDP, and SDP is better than general convex program. (*Hint:* use $z = (\mathbf{Re}(x), \mathbf{Im}(x)) \in \mathbb{R}^{2n}$ as the variable.)

4. (4 points) *Dual problems.* Consider the problem of projecting a point $a \in \mathbb{R}^n$ on the unit ball in ℓ_1-norm:

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|x-a\|_2^2 \\ \text{subject to} & \|x\|_1 \leq 1, \end{array}$$

with variables $x \in \mathbb{R}^n$. Derive the dual problem. (*Hint:* use the additional variable $y = x - a$.)

5. (4 points) *Semidefinite relaxation of nonconvex problem.*
A signal $\hat{s} \in \{-1, 1\}^n$ is sent through a noisy channel, and received as $y = H\hat{s} + w$, where $H \in \mathbb{R}^{m \times n}$, $y \in \mathbb{R}^m$, and w is the Gaussian noise. The maximum likelihood detection of the signal can be formulated as the following nonconvex problem:

$$\begin{array}{ll} \text{minimize} & \|y - Hs\|_2^2 \\ \text{subject to} & s_i^2 = 1, \; i = 1, \ldots, n. \end{array} \tag{1}$$

In the next series of questions, we explore the semidefinite relaxation (SDR) of the original nonconvex problem (1), and relationship between their dual problems.

(a) (1 point) Show that the original problem (1) can be rewritten in homogenous form:

$$\begin{array}{ll} \text{minimize} & x^T L x \\ \text{subject to} & x_i^2 = 1, \; i = 1, \ldots, n, \\ & x_{n+1} = 1. \end{array} \tag{2}$$

Specify L as a function of the original problem data H and y. (*Hint:* define $x = \begin{bmatrix} s \\ 1 \end{bmatrix}$.)

(b) (2 points) Derive the dual problem of the problem in homogenous form (2).

(c) (1 point) Now we consider the SDR of the nonconvex problem (2). We define a new variable $X = xx^T \in \mathbb{S}_+^n$. The problem in (2) is equivalent to

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(LX) \\ \text{subject to} & \mathbf{diag}(X) = 1_{n+1}, \\ & X \succeq 0, \\ & \mathbf{rank}(X) = 1, \end{array}$$

where 1_{n+1} is $(n+1)$-dimensional vector of 1's. By removing the nonconvex rank constraint, we have the following SDR:

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(LX) \\ \text{subject to} & \mathbf{diag}(X) = 1_{n+1}, \\ & X \succeq 0, \end{array} \tag{3}$$

which is a semidefinite program (SDP).

Derive the dual problem of the SDR (3) and show that it is the same as the dual of the original problem derived in part (b). (In other words, the SDR is the dual of the dual of the original nonconvex problem.)

(*Hint:* the Lagrangian multiplier associate with the constraint $X \succeq 0$ is $Z \succeq 0$, and we need to use the term $-\mathbf{tr}(ZX)$ in the Lagrangian.)