Convex Optimization

Lecture 8 - Applications in Smart Grids

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Today's Lecture

1. Generalized Inequalities and Semidefinite Programming
2. Overview of Smart Grids
3. Optimal Power Flow and Extensions

Outline

1. Generalized Inequalities and Semidefinite Programming
2. Overview of Smart Grids
3. Optimal Power Flow and Extensions

Proper Cones

a convex cone $K \subseteq \mathbb{R}^n$ is proper if:

- K is closed
- K has nonempty interior
- K is pointed (i.e., contains no line)

examples of proper cones:

- nonnegative orthant

$$K = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x_i \geq 0, i = 1, \ldots, n\}$$

- positive semidefinite cone

$$K = \mathbb{S}^n_+$$

Generalized Inequalities

generalized inequality defined by proper cone K:

$$x \preceq_K y \;\Leftrightarrow\; y - x \in K, \qquad x \prec_K y \;\Leftrightarrow\; y - x \in \mathrm{int} K$$

examples of generalized inequalities:

- component-wise inequality ($K = \mathbb{R}^n_+$):

$$x \preceq_{\mathbb{R}^n_+} y \;\Leftrightarrow\; x_i \leq y_i, \; i = 1, \ldots, n$$

- matrix inequality ($K = \mathbb{S}^n_+$):

$$X \preceq_{\mathbb{S}^n_+} Y \;\Leftrightarrow\; Y - X \text{ positive semidefinite}$$

subscripts usually dropped when $K = \mathbb{R}^n_+$ or $\mathbb{S}^n_+$

Some Properties

some properties are similar to $\leq$ on $\mathbb{R}$:

$$x \preceq_K y, \; u \preceq_K v \;\Rightarrow\; x + u \prec_K y + v$$

some properties are different:

- may not have a linear ordering:

  it is possible that $x \not\preceq_K y$ and $y \not\preceq_K x$

- may not have a minimum element for any subset S:

  may not exist x such that $x \preceq y, \; \forall y \in S$

Convexity With Respect To Generalized Inequalities

$f : \mathbb{R}^n \to \mathbb{R}^m$ is K-convex if $\operatorname{dom} f$ is convex and

$$f(\theta x + (1-\theta)y) \preceq_K \theta f(x) + (1-\theta)f(y)$$

for any $x, y \in \operatorname{dom} f$ and $\theta \in [0,1]$

examples:

- $f : \mathbb{S}^m \to \mathbb{S}^m,\ f(X) = X^2$ is $\mathbb{S}^m_+$-convex
    - proof: use the fact that $z^T X^2 z = \|Xz\|_2^2$ is convex in X

Convex Optimization With Generalized Inequalities

convex optimization with generalized inequality constraints:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \preceq_{K_i} 0, \;\; i = 1, \dots, m \\ & Ax = b \end{array}$$

where

- $f_0 : \mathbb{R}^n \to \mathbb{R}$ is convex
- $f_i : \mathbb{R}^n \to \mathbb{R}^{k_i}$ is K_i-convex

Semidefinite Program (SDP)

semidefinite program:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0 \\ & Ax = b \end{array}$$

where $F_i, G \in \mathbb{S}^k$

- inequality constraint called linear matrix inequality (LMI)
- can include multiple LMI constraints

$$x_1 \hat{F}_1 + x_2 \hat{F}_2 + \cdots + x_n \hat{F}_n + \hat{G} \preceq 0, \quad x_1 \tilde{F}_1 + x_2 \tilde{F}_2 + \cdots + x_n \tilde{F}_n + \tilde{G} \preceq 0$$

is equivalent to

$$x_1 \begin{bmatrix} \hat{F}_1 & 0 \\ 0 & \tilde{F}_1 \end{bmatrix} + x_2 \begin{bmatrix} \hat{F}_2 & 0 \\ 0 & \tilde{F}_2 \end{bmatrix} + \cdots x_n \begin{bmatrix} \hat{F}_n & 0 \\ 0 & \tilde{F}_n \end{bmatrix} + \begin{bmatrix} \hat{G} & 0 \\ 0 & \tilde{G} \end{bmatrix} \preceq 0$$

Semidefinite Program (SDP) - Standard Form

standard form semidefinite program:

$$\begin{aligned} \text{minimize} \quad & \text{tr}(CX) \\ \text{subject to} \quad & \text{tr}(A_i X) = b_i, \ \ i = 1, \ldots, p \\ & X \succeq 0 \end{aligned}$$

where

- optimization variable $X \in \mathbb{S}^n$
- $C, A_1, \ldots, A_p \in \mathbb{S}^n$
- $\text{tr}(\cdot)$ is the trace of a matrix:

$$\text{tr}(X) \triangleq \sum_{i=1}^{n} x_{ii}$$

- $\text{tr}(CX) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$

LP as SDP

linear program (LP):

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

equivalent SDP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \text{diag}(Ax - b) \preceq 0 \end{array}$$

diagonal matrix semidefinite $\Leftrightarrow$ each diagonal element nonnegative

SOCP as SDP

second-order cone program (SOCP):

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \;\; i = 1, \ldots, m \end{array}$$

equivalent SDP:

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \begin{bmatrix} \left(c_i^T x + d_i\right) I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{bmatrix} \succeq 0, \;\; i = 1, \ldots, m \end{array}$$

important fact in SDP:

$$\|A\|_2 \leq s \;\; \Leftrightarrow \;\; A^T A \preceq s^2 I \;\; \Leftrightarrow \begin{bmatrix} sI & A \\ A^T & sI \end{bmatrix} \succeq 0$$

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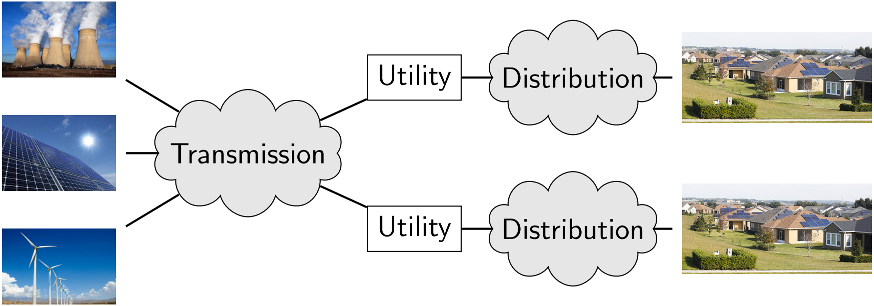
Traditional Power Systems → Smart Grid

traditional power systems:

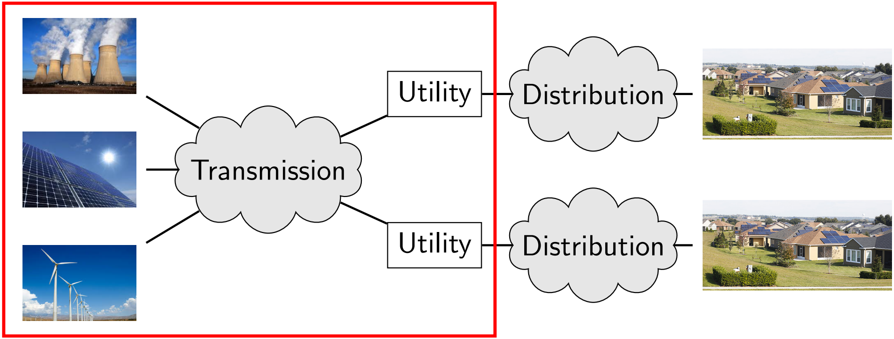


smart grid:

- integration of renewables
- deregulated electricity market
- coupling with other infrastructures

efficient optimization and computation is key!

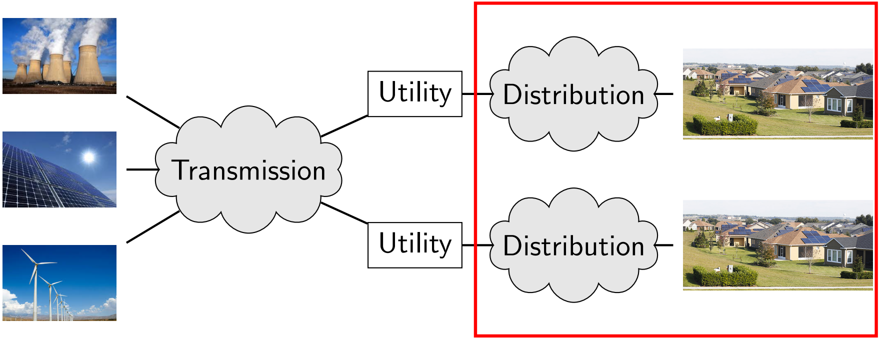
Transmission Grid



transmission grid:

- high voltage
- bulk power generators (e.g., coal, hydro-electric generators, wind farms)
- complicated topology

Distribution Grid

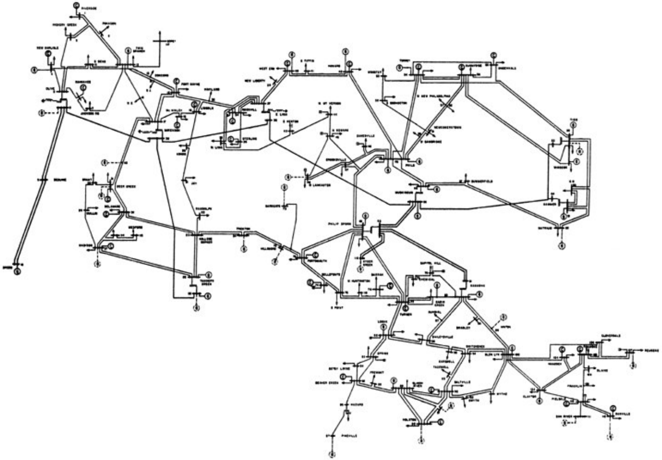


distribution grid:

- low voltage
- small power generators (e.g., residential solar panels)
- tree topology

Example Topology of Transmission Grids

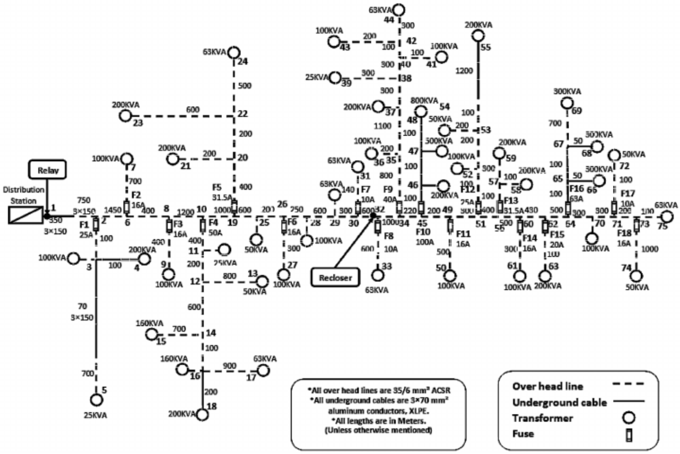
transmission grid (IEEE 118-bus system):



complicated with many cycles

Example Topology of Distribution Grids

distribution grid:



tree / radial networks

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Graph Model of The Grid

a power system is usually modeled by a (undirected) graph $(\mathcal{N}, \mathcal{E})$

- $\mathcal{N}$: set of nodes representing generator and/or load
- $\mathcal{E}$: set of edges representing transmission lines

key elements:

- (complex) power injection at node j: $s_j \in \mathbb{C}$
- admittance of line $(i, j) \in \mathcal{E}$: $y_{ij} \in \mathbb{C}$
  - usually $y_{ij} = y_{ji}$

Kirchhoff's law:

$$s_j = \sum_{k:(j,k)\in\mathcal{E}} y_{jk}^H V_j \left(V_j^H - V_k^H \right)$$

Optimal Power Flow

optimal power flow (vanilla version):

$$\begin{array}{ll} \text{minimize} & C(V) \\ \text{subject to} & \underline{v}_j \leq |V_j|^2 \leq \bar{v}_j, \; j \in \mathcal{N} \\ & \underline{s}_j \leq \displaystyle\sum_{k:(j,k)\in\mathcal{E}} y_{jk}^H V_j \left(V_j^H - V_k^H \right) \leq \bar{s}_j, \; j \in \mathcal{N} \end{array}$$

where

- $V \in \mathbb{C}^n$: the vector of voltages at each bus
- $C(V)$: cost of generation, loss of power in transmission, etc.
- voltage magnitude constraints: stability of transmission lines
- power injection constraints: physics of generators

usually solved by system operators:

- extremely important, solved every 5-15 minutes
- nonconvex

QCQP Formulation

cost is usually quadratic:

$$C(V) = V^H C V$$

admittance matrix $Y \in \mathbb{C}^{n\times n}$:

$$Y_{ij} = \begin{cases} \sum_{k:(i,k)\in\mathcal{E}} y_{ik} & \text{if } i = j \\ -y_{ij} & \text{if } i \neq j \text{ and } (i,j) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

Ohm's law: (I is the current injections to each bus)

$$I = YV$$

QCQP Formulation

power injections:

$$\begin{aligned} s_j &= V_j I_j^H = \left(e_j^H V\right)\left(I^H e_j\right) \\ &= \operatorname{tr}\left(e_j^H V V^H Y^H e_j\right) = \operatorname{tr}\left(\left(Y^H e_j e_j^H\right) V V^H\right) \triangleq \operatorname{tr}\left(Y_j V V^H\right) \\ &= \operatorname{tr}\left(V^H Y_j V\right) = V^H Y_j V \end{aligned}$$

similarly, $V_j V_j^H = V^H J_j V$, where $J_j = e_j e_j^H$

optimal power flow as nonconvex QCQP:

$$\begin{aligned} \text{minimize} \quad & V^H C V \\ \text{subject to} \quad & \underline{v}_j \le V^H J_j V \le \bar{v}_j, \ j \in \mathcal{N} \\ & \underline{s}_j \le V^H Y_j V \le \bar{s}_j, \ j \in \mathcal{N} \end{aligned}$$

Equivalent Formulation

observe:

$$V^H M V = \operatorname{tr}\left(M V V^H\right) = \operatorname{tr}(MW)$$

where $W \triangleq V V^H \in \mathbb{C}^{n \times n}$

equivalent problem:

$$\begin{array}{ll} \text{minimize} & \operatorname{tr}(CW) \\ \text{subject to} & \underline{v}_j \leq \operatorname{tr}(J_j W) \leq \bar{v}_j, \ j \in \mathcal{N} \\ & \underline{s}_j \leq \operatorname{tr}(Y_j W) \leq \bar{s}_j, \ j \in \mathcal{N} \\ & W \geq 0, \ \operatorname{rank}(W) = 1 \end{array}$$

only nonconvexity comes from $\operatorname{rank}(W) = 1$

Semidefinite Programming Relaxation

SDP relaxation by discarding rank constraint:

$$\begin{aligned} \text{minimize} \quad & \operatorname{tr}(CW) \\ \text{subject to} \quad & \underline{v}_j \le \operatorname{tr}(J_j W) \le \bar{v}_j, \; j \in \mathcal{N} \\ & \underline{s}_j \le \operatorname{tr}(Y_j W) \le \bar{s}_j, \; j \in \mathcal{N} \\ & W \ge 0 \end{aligned}$$

convex, can be efficiently solved

- solution to SDP relaxation: W_{sdp}; solution to OPF: $V^\star$
- if W_{sdp} is rank-1, then $W_{\text{sdp}} = V^\star (V^\star)^H$

when is relaxation exact? how tight is the relaxation?

Exactness and Tightness of Relaxation

relaxation is exact if the network is tree

rank of solution $W_{\text{sdp}} \leq$ treewidth of the network

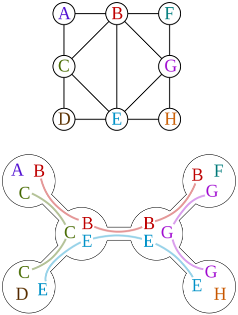
link exactness / tightness to the network topology

Tree Decomposition of Graph

tree decomposition of a graph:

- a tree with nodes $X_1, \ldots, X_m$, where X_i is subset of $\mathcal{N}$
  1. union of all sets X_i is $\mathcal{N}$
  2. if X_i and X_j both contain $k \in \mathcal{N}$, all nodes in the path between X_i and X_j contain k
  3. if $(k, \ell) \in \mathcal{N}$, there is a set X_i that contain k and ℓ

example:



Treewidth of Graph

tree decomposition is not unique

a trivial tree decomposition for any graph: tree with 1 node

width of a tree decomposition: size of largest set X_i minus one

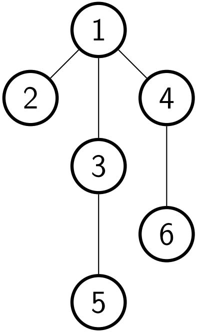
$$\max |X_i| - 1$$

treewidth of a graph:

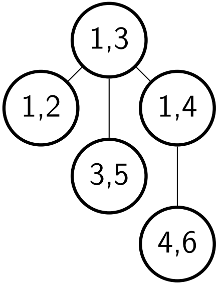
minimum width among all tree decompositions

Examples of Treewidth

treewidth of a tree is 1:



original graph

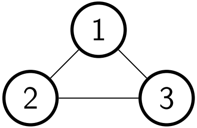


tree decomposition

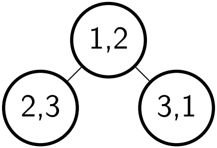
each subset X_i is an edge (i.e., two nodes) in the original tree

Examples of Treewidth

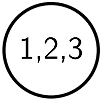
a fully-connected graph:



original graph



invalid tree decomposition



tree decomposition

for a fully-connected graph:

- unique tree decomposition: the trivial one
- treewidth: $|\mathcal{N}| - 1$

Power System State Estimation

power of electricity flow on each transmission line ℓ:

$$V^H H_\ell V = \operatorname{tr}(H_\ell W) \quad \text{according to physics}$$

measurements at a few selected lines:

$$z_\ell = \operatorname{tr}(H_\ell W) + \text{noise}$$

power system state estimation: find V that minimizes the estimation error

$$\begin{aligned} \text{minimize} \quad & \sum_{\ell \in \mathcal{L}} [z_\ell - \operatorname{tr}(H_\ell W)]^2 \\ \text{subject to} \quad & W \geq 0, \ \operatorname{rank}(W) = 1 \end{aligned}$$

Power System State Estimation

SDP relaxation:

$$\begin{aligned} \text{minimize} \quad & \sum_{\ell \in \mathcal{L}} x_\ell \\ \text{subject to} \quad & \begin{bmatrix} -x_\ell & z_\ell - \operatorname{tr}(H_\ell W) \\ z_\ell - \operatorname{tr}(H_\ell W) & -1 \end{bmatrix} \preceq 0 \\ & W \succeq 0 \end{aligned}$$