Convex Optimization

Lecture 8 - Applications in Wireless Communications

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Today's Lecture

1. Information Theory
2. Power Control
3. Beamforming

Outline

1. Information Theory
2. Power Control
3. Beamforming

Shannon Capacity of Discrete Memoryless Channels

discrete memoryless channel:

$$\text{input } X \longrightarrow \boxed{\text{channel } P} \longrightarrow \text{output } Y$$

- input: $X \in \{1, \ldots, n\}$
- output: $Y \in \{1, \ldots, m\}$
- channel transition matrix: $P \in \mathbb{R}^{m \times n}$ where

$$p_{ij} = \mathrm{prob}\,(Y = i | X = j)$$

Shannon Capacity of Discrete Memoryless Channels

Shannon capacity C (bits per second):

- information can be sent over the channel, with arbitrarily small error probability, at any rate less than C

Shannon's results tell us how to compute Shannon capacity:

- probability distribution of input X: $x \in \mathbb{R}^n$ where

$$x_j = \text{prob}\left(X = j\right)$$

- mutual information between input X and input Y:

$$I(X; Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_j p_{ij} \log_2 \frac{p_{ij}}{\sum_{k=1}^{n} x_k p_{ik}}$$

- channel capacity is

$$C = \sup_x I(X; Y)$$

Finding Shannon Capacity as Convex Program

rewrite the expression of mutual information:

$$
\begin{aligned}
I(X;Y) &= \sum_{i=1}^{m}\sum_{j=1}^{n} x_j p_{ij} \log_2 \frac{p_{ij}}{\sum_{k=1}^{n} x_k p_{ik}} \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} x_j p_{ij} \left(\log_2 p_{ij} - \log_2 \sum_{k=1}^{n} x_k p_{ik} \right) \\
&= \sum_{j=1}^{n} x_j \underbrace{\sum_{i=1}^{m} p_{ij} \log_2 p_{ij}}_{\triangleq c_j} - \sum_{i=1}^{m} \left(\underbrace{\sum_{j=1}^{n} x_j p_{ij}}_{\triangleq y_i} \right) \log_2 \sum_{k=1}^{n} x_k p_{ik} \\
&= \sum_{j=1}^{n} x_j c_j - \sum_{i=1}^{m} y_i \log_2 y_i
\end{aligned}
$$

Finding Shannon Capacity as Convex Program

find Shannon capacity:

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} x_j c_j - \sum_{i=1}^{m} y_i \log_2 y_i \\ \text{subject to} \quad & y_i = \sum_{j=1}^{n} x_j p_{ij}, i = 1, \ldots, m \\ & \sum_{j=1}^{n} x_j = 1 \\ & x_j \geq 0, \; j = 1, \ldots, n \end{aligned}$$

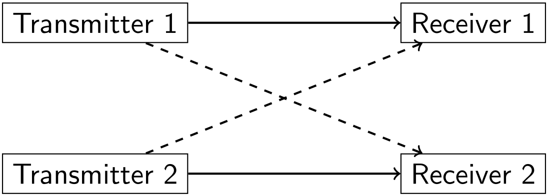
see `http://cvxr.com/cvx/examples/`

- "**Figures, examples, and exercises from the book**"
- "**Chapter 4**"
- "Exercise 4.57: Capacity of a communication channel"

Outline

Wireless Communication Systems

a wireless communication system with 2 transmitter-receiver pairs:



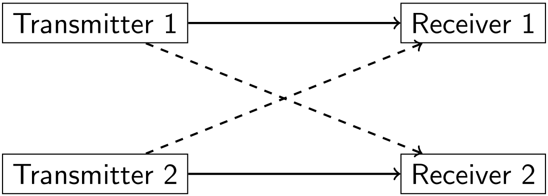
in general, we have

- n pairs of transmitters and receivers
- matrix of path gains $G \in \mathbb{R}^{n \times n}$ where

$$g_{ij} : \text{path gain from transmitter } j \text{ to receiver } i$$

- noise power at receiver i: σ_i

Key Elements



key elements:

- transmit power level at transmitter i: p_i
- signal power at receiver i: $S_i = p_i g_{ii}$
- interference power at receiver i: $I_i = \sum_{j \neq i} g_{ij} p_j$
- signal-to-interference-and-noise ratio (SINR) for the ith pair:

$$\frac{S_i}{I_i + \sigma_i}$$

Power Control for SINR Satisfaction

power control for SINR satisfaction

- objective: minimize the total transmit power $\sum_{i=1}^{n} p_i$
- quality of service constraints:

$$\frac{g_{ii} p_i}{\sum_{j \neq i} g_{ij} p_j + \sigma_i} \geq \gamma_i$$

convex problem formulation (equivalent to a LP):

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{n} p_i \\ \text{subject to} & \dfrac{g_{ii} p_i}{\sum_{j \neq i} g_{ij} p_j + \sigma_i} \geq \gamma_i, \;\; i = 1, \ldots, n \end{array}$$

Power Control for SINR Maximization

power control for SINR maximization:

- objective: maximize the minimum SINR

$$\text{maximize} \quad \min_{i=1,\ldots,n} \frac{S_i}{I_i + \sigma_i}$$

- maximum transmit power constraints: $p_i \in [0, P_i^{\max}]$

additional constraints (do not change the nature of the problem):

- group power supply constraints:
  - subsets of transmitters $K_1, \ldots, K_m$
  - transmitters in the same subset share the same power supply

$$\sum_{k \in K_\ell} p_k \leq P_\ell^{\text{gp}}, \quad \ell = 1, \ldots, m$$

- maximum receive power constraints:

$$\sum_{k=1}^{n} g_{ik} p_k \leq P_i^{\text{rc}}, \quad i = 1, \ldots, n$$

Formulation as Quasiconvex Optimization Problem

express SINR as:

$$\frac{S_i}{I_i + \sigma_i} = \frac{g_{ii} p_i}{\sum_{j \neq i} g_{ij} p_j + \sigma_i}$$

generalized linear-fractional program:

$$\begin{array}{ll} \text{maximize} & \min\limits_{i=1,\ldots,n} \dfrac{g_{ii} p_i}{\sum_{j \neq i} g_{ij} p_j + \sigma_i} \\ \text{subject to} & 0 \leq p_i \leq P_i^{\max}, \ \ i = 1, \ldots, n \end{array}$$

quasiconvex optimization

Solve The Quasiconvex Optimization Problem

for given t, a convex feasibility problem:

$$\begin{array}{ll} \text{maximize} & 0 \\ \text{subject to} & \min\limits_{i=1,\ldots,n} \dfrac{g_{ii}p_i}{\sum_{j\neq i} g_{ij}p_j + \sigma_i} \geq t, \\ & 0 \leq p_i \leq P_i^{\max}, \ \ i = 1, \ldots, n \end{array}$$

equivalent formulation (actually a LP)

$$\begin{array}{ll} \text{maximize} & 0 \\ \text{subject to} & \dfrac{g_{ii}p_i}{\sum_{j\neq i} g_{ij}p_j + \sigma_i} \geq t, \ \ i = 1, \ldots, n \\ & 0 \leq p_i \leq P_i^{\max}, \ \ i = 1, \ldots, n \end{array}$$

use bisection method to find the maximum t

Power Control for Throughput Maximization

throughput (under certain conditions):

$$\log_2(1+\mathrm{SINR}) = \log_2\left(1 + \frac{g_{ii}p_i}{\sum_{j\neq i} g_{ij}p_j + \sigma_i}\right)$$

normalize so that $g_{ii} = 1, \ i = 1, \ldots, n$

power control for throughput maximization:

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{n} \log_2\left(1 + \frac{p_i}{\sum_{j\neq i} g_{ij}p_j + \sigma_i}\right) \\ \text{subject to} & p_i \geq 0, \ i = 1, \ldots, n \\ & \sum_{i=1}^{n} p_i = 1 \end{array}$$

nonconvex (due to the appearance of p in the denominator)

Reformulation as Convex Optimization

rewrite the expression of throughput:

$$\begin{aligned}
\log_2\left(1+\frac{p_i}{\sum_{j\neq i} g_{ij}p_j+\sigma_i}\right) &= \log_2\left(\frac{\sum_{j=1}^n g_{ij}p_j+\sigma_i}{\sum_{j=1}^n g_{ij}p_j-p_i+\sigma_i}\right)\\
&= \log_2\left(\frac{\sum_{j=1}^n g_{ij}p_j+\sum_{j=1}^n p_j\sigma_i}{\sum_{j=1}^n g_{ij}p_j-p_i+\sum_{j=1}^n p_j\sigma_i}\right)\\
&= \log_2\left(\frac{\sum_{j=1}^n h_{ij}p_j}{\sum_{j=1}^n h_{ij}p_j-p_i}\right)
\end{aligned}$$

where $H = G + \sigma \cdot 1^T$, namely $h_{ij} = g_{ij} + \sigma_i$

change of variable: $y = Hp$, namely $y_i = \sum_{j=1}^n h_{ij}p_j$

under some conditions (e.g., $g_{ii} > \sum_{j\neq i} g_{ij}$), we have

- H invertible with $H^{-1} = I - C$
- C has nonnegative elements

Reformulation as Convex Optimization

$$p = (I - C)y \Rightarrow p_i = (1 - c_{ii})y_i - \sum_{j \neq i} c_{ij} y_j = y_i - c_i^T y$$

convex reformulation:

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{n} \log_2 \left(\frac{y_i}{c_i^T y} \right) \\ \text{subject to} & y_i - c_i^T y \geq 0, \ \ i = 1, \ldots, n \\ & \sum_{i=1}^{n} \left(y_i - c_i^T y \right) = 1 \end{array}$$

convex (need to show the Hessian is negative semidefinite)

interpretation: the problem is “easy” under weak interference (i.e., $g_{ii} > \sum_{j \neq i} g_{ij}$)

Power Control Under No Multi-User Interference

throughput under no interference:

$$\log_2\left(1+\frac{g_{ii}p_i}{\sigma_i}\right)=\log_2\left(\frac{\sigma_i}{g_{ii}}+p_i\right)+\log_2\frac{g_{ii}}{\sigma_i}$$

power control for throughput maximization:

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{n}\log_2\left(\alpha_i+p_i\right) \\ \text{subject to} & p_i\geq 0,\ \ i=1,\dots,n \\ & \sum_{i=1}^{n}p_i=1 \end{array}$$

where $\alpha_i=\frac{\sigma_i}{g_{ii}}$

KKT Conditions

Lagrange multipliers associated with $p_i \geq 0$: λ_i

Lagrange multiplier associated with $\sum_{i=1}^{n} p_i = 1$: ν

KKT conditions:

$$
\begin{aligned}
\frac{1}{\alpha_i + p_i} + \lambda_i - \nu &= 0, \; i = 1, \ldots, n \\
p_i &\geq 0, \; i = 1, \ldots, n \\
\sum_{i=1}^{n} p_i &= 1, \\
\lambda_i &\geq 0, \; i = 1, \ldots, n \\
\lambda_i p_i &= 0, \; i = 1, \ldots, n
\end{aligned}
$$

KKT Conditions

KKT conditions after eliminating λ:

$$
\begin{aligned}
\frac{1}{\alpha_i + p_i} &\leq \nu, \; i = 1, \ldots, n \\
p_i &\geq 0, \; i = 1, \ldots, n \\
\sum_{i=1}^{n} p_i &= 1, \\
\left(\nu - \frac{1}{\alpha_i + p_i}\right) p_i &= 0, \; i = 1, \ldots, n
\end{aligned}
$$

- if $\nu < \frac{1}{\alpha_i}$, we have $p_i > 0 \Rightarrow p_i = \frac{1}{\nu} - \alpha_i$
- if $\nu \geq \frac{1}{\alpha_i}$, we have $p_i = 0$

Optimal Solution as "Water-Filling"

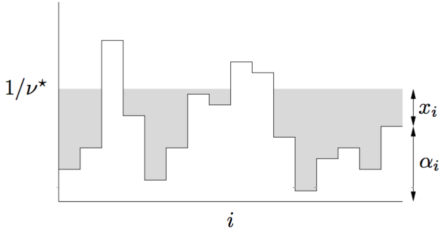
optimal solution:

$$p_i^\star = \begin{cases} \frac{1}{\nu^\star} - \alpha_i & \nu^\star < \frac{1}{\alpha_i} \\ 0 & \nu^\star \geq \frac{1}{\alpha_i} \end{cases}$$

where $\nu^\star$ is determined by solving the following equation:

$$\sum_{i=1}^{n} \max\left\{0, \frac{1}{\nu^\star} - \alpha_i\right\} = 1$$

"water-filling" interpretation:



Joint Power Control and Bandwidth Allocation

transmitter-receiver pairs use frequency-division multiple access

- different pairs use non-overlapping frequency bands
- no interference across pairs
- need to determine bandwidth allocation

throughput:

$$R_i(p_i, W_i) = W_i \log_2 \left(1 + \frac{g_{ii} p_i}{\sigma_i W_i}\right)$$

- W_i: bandwidth allocated to pair i
- σ_i: noise power per unit bandwidth at receiver i

Joint Power Control and Bandwidth Allocation

joint power control and bandwidth allocation:

$$
\begin{array}{ll}
\text{maximize} & \sum_{i=1}^{n} W_i \log_2 \left(1 + \frac{g_{ii} p_i}{\sigma_i W_i}\right) \\
\text{subject to} & p_i \geq 0, \ \ i = 1, \ldots, n \\
& \sum_{i=1}^{n} p_i = 1 \\
& W_i \geq 0, \ \ i = 1, \ldots, n \\
& \sum_{i=1}^{n} W_i = 1
\end{array}
$$

convex optimization

- objective is concave (as a perspective function)

Outline

Radiation Pattern of Antenna Array

an antenna array with n elements in 2-D space

position of kth element: (x_k, y_k)

output of the array at direction θ

$$G(\theta) = \sum_{k=1}^{n} w_k e^{\mathbf{i}(x_k \cos\theta + y_k \sin\theta)}$$

w_k is the weight of (or current fed into) kth element

share some similarity to filter design

Sidelobe Minimization for Directional Antenna

sidelobe minimization for directional antenna:

$$\begin{array}{ll} \text{minimize} & \max_{|\theta-\theta^{\text{tar}}|\geq\Delta} |G(\theta)| \\ \text{subject to} & G(\theta^{\text{tar}}) = 1 \end{array}$$

- θ^{tar}: target direction
- Δ: beamwidth
- convex optimization (but need discretization)

smallest beamwidth Δ can be determined by bisection method

Limitations on Arbitrary Array Topology

problem is nonconvex under these objectives or constraints:

- bound on ripple effect:

$$1/\delta_1 \leq |G(\theta)| \leq \delta_1, \quad \left|\theta - \theta^{\mathsf{tar}}\right| \leq \Delta$$

- maximize gains at target direction:

$$\text{maximize} \quad \left|G(\theta^{\mathsf{tar}}\right|$$

Uniform Linear Array

uniform linear array: elements on a line with equal distances:

- all elements on a line
- same distance d between neighboring elements

output of uniform linear array at direction θ:

$$G(\theta) = \sum_{k=1}^{n} w_k e^{\mathbf{i}(k-1)d\cos\theta}$$

Uniform Linear Array

rewrite the expression of output:

$$
\begin{aligned}
|G(\theta)|^2 &= \left[\sum_{k=1}^{n} w_k e^{\mathbf{i}(k-1)d\cos\theta}\right]\cdot\left[\sum_{k=1}^{n} w_k^* e^{-\mathbf{i}(k-1)d\cos\theta}\right] \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} w_j w_k^* e^{\mathbf{i}(j-1)d\cos\theta} e^{-\mathbf{i}(k-1)d\cos\theta} \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} w_j w_k^* e^{\mathbf{i}(j-k)d\cos\theta} \\
&= \sum_{j=1}^{n}\sum_{\ell=j-1}^{j-n} w_j w_{j-\ell}^* e^{\mathbf{i}\ell d\cos\theta} \\
&= \sum_{\ell=-(n-1)}^{(n-1)}\ \sum_{m=\max\{\ell+1,1\}}^{\min\{\ell+n,n\}} w_j w_{j-\ell}^* e^{\mathbf{i}\ell d\cos\theta}
\end{aligned}
$$

Uniform Linear Array

define the autocorrelation of w

$$r_\ell = \sum_{m=\max\{\ell+1,1\}}^{\min\{\ell+n,n\}} w_j w^*_{j-\ell}$$

for $\ell = -(n-1), \ldots, 0, \ldots, (n-1)$

the output is:

$$|G(\theta)|^2 = \sum_{\ell=-(n-1)}^{(n-1)} r_\ell \cdot e^{\mathbf{i}\ell d \cos\theta}$$

find $r \in \mathbb{R}^{2n-1}$, then use spectral factorization to recover w