Convex Optimization

Lecture 7 - Applications in Signal Processimg

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Fall 2017

Today's Lecture

1. Statistical Estimation
2. Hypothesis Testing
3. Filter Design

Outline

1. Statistical Estimation

2. Hypothesis Testing

3. Filter Design

Parametric Distribution Estimation

$p_x(y)$: $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$

- as a function of y: probability distribution of y, indexed by x
- as a function of x: likelihood function for fixed y

parametric distribution estimation:

- given y, choose the "most likely" distribution $p_x(\cdot)$ (i.e., choose x)

Maximum Likelihood Estimation

maximum likelihood estimation:

$$\text{maximize} \quad p_x(y)$$

with optimization variable x

log-likelihood function: (more like to be concave)

$$\ell(x) = \log p_x(y)$$

equivalent formulation:

$$\text{maximize} \quad \ell(x) = \log p_x(y)$$

with optimization variable x

Linear Measurements With IID Noise

linear measurement model:

$$y_i = a_i^T x + v_i, \quad i = 1, \ldots, m$$

- $x_i \in \mathbb{R}^n$: parameters to be estimated
- $y_i \in \mathbb{R}$: measurement
- $v_i \in \mathbb{R}$: noise

we assume

- v_i are independent, identically distributed (IID)
- v_i has probability density function $p(\cdot)$

log-likelihood function:

$$\ell(x) = \log p_x(y) = \log \prod_{i=1}^{m} p\left(y_i - a_i^T x\right) = \sum_{i=1}^{m} \log p\left(y_i - a_i^T x\right)$$

Linear Measurements With IID Gaussian Noise

Gaussian noise with zero mean and variance σ^2:

$$p(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2\sigma^2}}$$

log-likelihood function:

$$\begin{aligned} \ell(x) &= -\frac{m}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{m}\left(y_i - a_i^T x\right)^2 \\ &= -\frac{m}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\|Ax - y\|_2^2 \end{aligned}$$

maximum likelihood estimation:

$$\text{minimize} \quad \|Ax - y\|_2^2$$

Linear Measurements With IID Laplacian Noise

Laplacian noise with zero mean and variance σ^2:

$$p(z) = \frac{1}{2a} e^{-\frac{|z|}{a}}$$

with $a > 0$

log-likelihood function:

$$\begin{aligned} \ell(x) &= -m \log(2a) - \frac{1}{a} \sum_{i=1}^{m} \left| y_i - a_i^T x \right| \\ &= -m \log(2a) - \frac{1}{a} \|Ax - y\|_1 \end{aligned}$$

maximum likelihood estimation:

$$\text{minimize} \quad \|Ax - y\|_1$$

Linear Measurements With IID Uniform Noise

uniformly distributed noise on $[-a, a]$:

$$p(z) = \begin{cases} \frac{1}{2a} & \text{if } z \in [-a, a] \\ 0 & \text{otherwise} \end{cases}$$

log-likelihood function:

$$\begin{aligned} \ell(x) &= \begin{cases} -m \log(2a) & \text{if } \left|y_i - a_i^T x\right| \leq a, \ \ i = 1, \ldots, m \\ -\infty & \text{otherwise} \end{cases} \\ &= \begin{cases} -m \log(2a) & \text{if } \|Ax - y\|_\infty \leq a \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

maximum likelihood estimation:

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & \|Ax - y\|_\infty \leq a \end{array}$$

a feasibility problem

Counting Problems With Poisson Distribution

measurement y is nonnegative integer with Poisson distribution:

$$\text{prob}(y = k) = \frac{e^{-\mu}\mu^k}{k!}$$

- e.g., # of cars passing an intersection, # of traffic accidents
- μ is the average number per unit time

assume $\mu = a^T u + b$

- $u \in \mathbb{R}^n$: explanatory variables
  - e.g., $u = (\text{total traffic flow}, \text{rainfall}, \text{peak hours or not}) \in \mathbb{R}^3$
- $a \in \mathbb{R}^n, b \in \mathbb{R}$: model parameters to be estimated

try to estimate how traffic accidents depend on various variables

Counting Problems With Poisson Distribution

m measurements: $(u_i, y_i),\ i = 1, \ldots, m$

likelihood function:

$$\prod_{i=1}^{m} \frac{(a^T u_i + b)^{y_i} e^{-(a^T u_i + b)}}{y_i!}$$

log-likelihood function:

$$\ell(x) = \sum_{i=1}^{m} \left[y_i \log(a^T u_i + b) - (a^T u_i + b) - \log(y_i!) \right]$$

maximum likelihood estimation:

$$\text{maximize} \quad \sum_{i=1}^{m} \left[y_i \log(a^T u_i + b) - (a^T u_i + b) \right]$$

with optimization variables $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$

Nonparametric Distribution Estimation

a random variable X with values in finite set $\{\alpha_1, \ldots, \alpha_n\}$

- distribution of X characterized by $p \in \mathbb{R}^n$,
  where $\mathrm{prob}(X = \alpha_k) = p_k, \;\; k = 1, \ldots, n$

nonparametric distribution estimation:

- estimate the distribution p
- base on prior information, observations, measurements, etc.

Prior Information

prior information $\Rightarrow$ constraints on p

examples of prior information that result in linear constraints:

- mean: $\mathbf{E}(X) = \alpha \Rightarrow \sum_{i=1}^{n} \alpha_i p_i = \alpha$
- second moment: $\mathbf{E}(X^2) = \beta \Rightarrow \sum_{i=1}^{n} \alpha_i^2 p_i = \beta$
- $\text{prob}(X \geq 0) \leq 0.2 \Rightarrow \sum_{i:\alpha_i \geq 0} p_i \leq 0.2$

examples of prior information that result in nonlinear constraints:

- variance: (concave in p)

$$\mathbf{var}(X) = \mathbf{E}(X^2) - [\mathbf{E}(X)]^2 = \sum_{i=1}^{n} \alpha_i^2 p_i - \sum_{i=1}^{n} (\alpha_i p_i)^2$$

- Kullback-Leiber divergence from distribution q: (convex in p)

$$\sum_{i=1}^{n} p_i \log(p_i / q_i)$$

Maximum Likelihood Estimation

prior information represented by $p \in \mathcal{P}$

- $\mathcal{P}$ results from the prior information discussed earlier
- $\mathcal{P} \subseteq \{p : 1^T p = 1, p \geq 0\}$

observations:

- N independent samples
- the number of samples with value α_i is k_i

log-likelihood function

$$\ell(p) = \log \prod_{i=1}^{n} p_i^{k_i} = \sum_{i=1}^{n} k_i \log(p_i)$$

maximum likelihood estimation:

$$\begin{array}{ll} \text{maximize} & \ell(p) = \sum_{i=1}^{n} k_i \log(p_i) \\ \text{subject to} & p \in \mathcal{P} \end{array}$$

Other Problems

the distribution with the minimum expected value of a function:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} f(\alpha_i) p_i \\ \text{subject to} & p \in \mathcal{P} \end{array}$$

where $f(\cdot)$ can be any function (even nonconvex)

the distribution with minimum K-L divergence from q:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} p_i \log(p_i/q_i) \\ \text{subject to} & p \in \mathcal{P} \end{array}$$

- $q = \frac{1}{n}\mathbf{1}$: find the most random distribution

Outline

Hypothesis Testing

a random variable X with values in finite set $\{\alpha_1, \ldots, \alpha_n\}$

- distribution of X parameterized by $\theta \in \{\theta_1, \ldots, \theta_m\}$
- matrix $P \in \mathbb{R}^{n \times m}$ with

$$p_{ij} = \mathrm{prob}\,(X = \alpha_i | \theta = \theta_j)$$

observe samples of X, then estimate θ

- θ: hypothesis $\Rightarrow$ hypothesis testing
- θ: events $\Rightarrow$ event detection

note the difference from nonparametric distribution estimation:
P is known in hypothesis testing

Detector

detector: a random variable $\hat{\theta}$ with distribution depending on X

$T \in \mathbb{R}^{m\times n}$ with

$$t_{ji} = \text{prob}\left(\hat{\theta} = \theta_j | X = \alpha_i\right)$$

- when observe α_i, the detector give $\hat{\theta} = \theta_j$ with probability t_{ji}
- ith column of T, t_i: probability distribution of $\hat{\theta}$ given α_i

T must satisfy

$$t_i \succeq 0, \quad \mathbf{1}^T t_i = 1, \; i = 1, \ldots, n$$

Detection Probability Matrix

detection probability matrix: $D = TP \in \mathbb{R}^{m \times m}$, where

$$\begin{aligned} D_{ji} &= (TP)_{ji} \\ &= \sum_{k=1}^{n} \text{prob}\left(\hat{\theta} = \theta_j | X = \alpha_k\right) \text{prob}\left(X = \alpha_k | \theta = \theta_i\right) \\ &= \text{prob}\left(\hat{\theta} = \theta_j | \theta = \theta_i\right) \end{aligned}$$

- detection probabilities denoted by $P^d \in \mathbb{R}^m$ with

$$P_i^d = D_{ii} = \text{prob}\left(\hat{\theta} = \theta_i | \theta = \theta_i\right)$$

- error probabilities denoted by $P^e \in \mathbb{R}^m$ with

$$P_i^e = \sum_{j \neq i} D_{ji} = \text{prob}\left(\hat{\theta} \neq \theta_i | \theta = \theta_i\right)$$

P^d and P^e are linear in detector T

Optimal Detector Design – Minimax Detector

minimax detector design:

$$\begin{array}{ll} \text{minimize} & \max_j P_j^e \\ \text{subject to} & t_i \geq 0, \mathbf{1}^T t_i = 1, i = 1, \ldots, n \end{array}$$

minimize the worst-case error probability

can add many constraints:

- lower bounds on detection probabilities:

$$P_i^d = D_{ii} \geq L_i$$

- upper bounds on error probabilities:

$$D_{ji} \leq U_{ji}$$

Optimal Detector Design – Bayes Detector

suppose that there is a prior distribution for the hypotheses $q \in \mathbb{R}^m$

$$q_i = \text{prob}\,(\theta = \theta_i)$$

Bayes detector design:

$$\begin{array}{ll} \text{minimize} & q^T P^e \\ \text{subject to} & t_i \geq 0, 1^T t_i = 1, i = 1, \ldots, n \end{array}$$

minimize the expected error probability

Robust Detector Design – Robust Minimax Detector

suppose that P is not known exactly but $P \in \mathcal{P}$

robust (worst-case) minimax detector design:

$$\begin{array}{ll} \text{minimize} & \max_j \ \sup_{p \in P} P_j^e \\ \text{subject to} & t_i \geq 0, \mathbf{1}^T t_i = 1, i = 1, \ldots, n \end{array}$$

where $P_j^e = 1 - D_{jj}$

Outline

FIR Filters

finite impulse response (FIR) filters:

$$y(t) = \sum_{\tau=0}^{n-1} h_\tau u(t-\tau), \quad t \in \mathbb{Z}$$

- u: input signal/sequence
- y: output signal/sequence
- h_i: filter coefficients
- n: filter length

Frequency Response

frequency response:

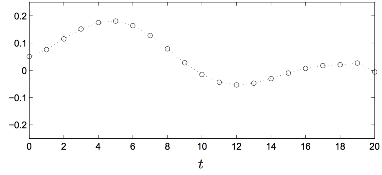
$$\begin{aligned} H(\omega) &= h_0 + h_1 e^{-j\omega} + \cdots + h_{n-1} e^{-j(n-1)\omega} \\ &= \sum_{t=0}^{n-1} h_t e^{-j(t-1)\omega} \end{aligned}$$

- $H(\omega + 2\pi) = H(\omega)$, $H(\omega + \pi) = -H(\omega)$
- need to specify $H(\cdot)$ only in $[0, \pi]$

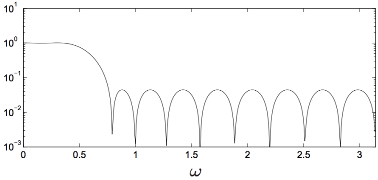
filter design: choose h so that h and H satisfy certain specifications

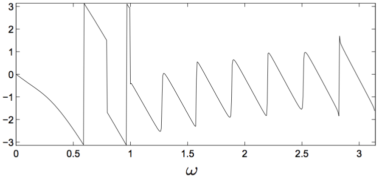
Frequency Response - Example

FIR filter $h(t)$ of length $n = 21$:



frequency response $H(\omega)$:





magnitude $|H(\omega)|$ phase $\angle H(\omega)|$

Chebychev Design

Chebychev design:

$$\text{minimize} \quad \max_{\omega \in [0,\pi]} |H(\omega) - H_{\text{des}}(\omega)|$$

with optimization variables $h(0), \ldots, h(n-1)$

- H_{des}: desired frequency response
- convex optimization (may not be easy to solve)

relaxation:

$$\text{minimize} \quad \max_{k=1,\ldots,m} |H(\omega_k) - H_{\text{des}}(\omega_k)|$$

with optimization variables $h(0), \ldots, h(n-1)$

- close to the desired frequency response at m sample points

Chebychev Design

Chebychev design as SOCP:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \|A_k h - b_k\| \leq t, \;\; k = 1, \ldots, m \end{array}$$

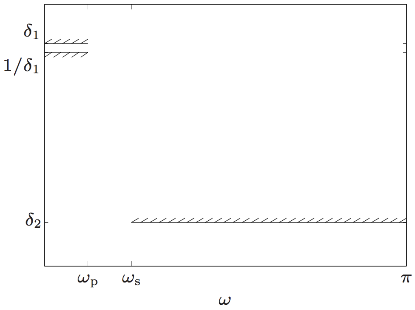
where

$$A_k = \begin{bmatrix} 1 & \cos\omega_k & \cdots & \cos(n-1)\omega_k \\ 0 & \sin\omega_k & \cdots & \sin(n-1)\omega_k \end{bmatrix}$$

$$b_k = \begin{bmatrix} \mathrm{Re}\left(H_{\mathrm{des}}(\omega_k)\right) \\ \mathrm{Im}\left(H_{\mathrm{des}}(\omega_k)\right) \end{bmatrix}$$

Lowpass Filter Design

lowpass filter design:



- low frequency $[0, \omega_p]$: magnitude within $[1/\delta_1, \delta_1]$

$$1/\delta_1 \leq |H(\omega)| \leq \delta_1, \; 0 \leq \omega \leq \omega_p$$

- high frequency $[\omega_s, \pi]$: magnitude below δ_2

$$|H(\omega)| \leq \delta_2, \; \omega_s \leq \omega \leq \pi$$

Lowpass Filter Design

samples at frequencies $\omega_1, \ldots, \omega_m$

minimum stopband magnitude:

$$
\begin{array}{ll}
\text{minimize} & \delta_2 \\
\text{subject to} & 1/\delta_1 \leq |H(\omega_k)| \leq \delta_1, \ 0 \leq \omega_k \leq \omega_p \\
& |H(\omega_k)| \leq \delta_2, \ \omega_s \leq \omega_k \leq \pi
\end{array}
$$

nonconvex due to $1/\delta_1 \leq |H(\omega_k)|$

Lowpass Filter Design – Linear Phase Filters

consider filter h such that

- $n = 2N + 1$ is odd
- h is symmetric: $h_t = h_{n-1-t}, \; t = 0, \ldots, n-1$

$$\begin{aligned} H(\omega) &= h_0 + h_1 e^{-j\omega} + \cdots + h_{n-1} e^{-j(n-1)\omega} \\ &= h_N e^{-jN\omega} + \sum_{t=0}^{N-1} h_t e^{-jt\omega} + h_{n-1-t} e^{-j(n-1-t)\omega} \\ &= h_N e^{-jN\omega} + \sum_{t=0}^{N-1} h_t \left(e^{-jt\omega} + e^{-j(2N-t)\omega} \right) \\ &= e^{-jN\omega} \left[h_N + \sum_{t=0}^{N-1} h_t \left(e^{-j(t-N)\omega} + e^{-j(N-t)\omega} \right) \right] \\ &= e^{-jN\omega} \left(h_N + \sum_{t=0}^{N-1} 2 h_t \cos(t-N)\omega \right) \end{aligned}$$

Lowpass Filter Design – Linear Phase Filters

linear phase filter:

- $H(\omega) = e^{-jN\omega}\tilde{H}(\omega)$
- phase $N\omega$ is linear in frequency
- magnitude is $\tilde{H}(\omega)$
- $\tilde{H}(\omega)$ is real and linear in h

minimum stopband magnitude:

$$\begin{array}{ll} \text{minimize} & \delta_2 \\ \text{subject to} & 1/\delta_1 \leq \tilde{H}(\omega_k) \leq \delta_1, \; 0 \leq \omega_k \leq \omega_p \\ & \tilde{H}(\omega_k) \leq \delta_2, \; \omega_s \leq \omega_k \leq \pi \end{array}$$

linear program

Lowpass Filter Design – Linear Phase Filters

minimum passband fluctuation:

$$\begin{array}{ll} \text{minimize} & \delta_1 \\ \text{subject to} & \delta_1 \geq 1 \\ & 1/\delta_1 \leq \tilde{H}(\omega_k) \leq \delta_1, \; 0 \leq \omega_k \leq \omega_p \\ & \tilde{H}(\omega_k) \leq \delta_2, \; \omega_s \leq \omega_k \leq \pi \end{array}$$

convex optimization (but not LP)

Lowpass Filter Design – Convex Reformulation

suppose that we consider general filters (not linear phase)

change of variables: autocorrelation coefficients:

$$r_t = \sum_{\tau} h_\tau h_{\tau+t}$$

where $h_t = 0$ for $t < 0$ and $t \geq n$

- $r_t = r_{-t}$ and $r_t = 0$ for $|t| \geq n$
- need to specify $r = (r_0, \ldots, r_{n-1}) \in \mathbb{R}^n$
- Fourier transform

$$R(\omega) = \sum_{\tau} r_\tau e^{-j\tau\omega} = r_0 + \sum_{t=1}^{n-1} 2 r_t \cos \omega t = |H(\omega)|^2$$

convex constraints: $L(\omega)^2 \leq R(\omega) \leq U(\omega)^2$