Convex Optimization

Lecture 4 - Convex Optimization Problems

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Today's Lecture

1. Basic Concepts
2. Operations That Produce Equivalent Problems
3. Important Examples
4. Quasiconvex Optimization Problems

Outline

Optimization Problems

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \ i = 1, \ldots, m \\ & h_i(x) = 0, \ i = 1, \ldots, p \end{array}$$

- $x \in \mathbb{R}^n$: optimization variables
- $f_0 : \mathbb{R}^n \to \mathbb{R}$: objective/cost function
- $f_i : \mathbb{R}^n \to \mathbb{R}, \ i = 1 \ldots, m$: inequality constraint functions
- $h_i : \mathbb{R}^n \to \mathbb{R}, \ i = 1 \ldots, p$: equality constraint functions

The problem is called unconstrained if there are no constraints.

Basic Terminology

domain: $\mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom} f_i \ \cap \ \bigcap_{i=1}^{p} \mathbf{dom} h_i$

- $x \in \mathcal{D}$ are implicit constraints
- example:

$$\text{minimize } f_0(x) = -\sum_{i=1}^{k} \log(b_i - a_i^T x)$$

an unconstrained problem with implicit constraints $a_i^T x < b_i$

feasible point: a point $x \in \mathcal{D}$ is *feasible* if it satisfies all the constraints $f_i(x) \le 0, \ i = 1, \ldots, m$ and $h_i(x) = 0, \ i = 1, \ldots, p$

feasible problem: the problem is *feasible* if there exists at least one feasible point, and *infeasible* otherwise

feasible set: the set of all feasible points

Optimality

the optimal value $p^\star$ is

$$p^\star = \inf\left\{f_0(x) | f_i(x) \leq 0,\ i = 1, \dots, m,\ h_i(x) = 0,\ i = 1, \dots, p\right\}$$

- $p^\star = \infty$: the problem is *infeasible*
- $p^\star = -\infty$: the problem is *unbounded below*

$x^\star$ is an optimal point, or solves the problem, if $x^\star$ is feasible and $f_0(x^\star) = p^\star$

a feasible point x is ϵ-suboptimal if $f_0(x) \leq p^\star + \epsilon$

x is locally-optimal if there exists $R > 0$ such that x solves

$$\begin{array}{ll} \text{minimize} & f_0(z) \\ \text{subject to} & f_i(z) \leq 0,\ i = 1, \dots, m \\ & h_i(z) = 0,\ i = 1, \dots, p \\ & \|z - x\|_2 \leq R \end{array}$$

Examples

Unconstrained problems with variable $x \in \mathbb{R}$:

- $f_0(x) = 1/x,\ \mathbf{dom} f_0 = \mathbb{R}_{++}$: $p^\star = 0$, no optimal point
- $f_0(x) = -\log x,\ \mathbf{dom} f_0 = \mathbb{R}_{++}$: $p^\star = -\infty$, unbounded below, no optimal point
- $f_0(x) = x \log x,\ \mathbf{dom} f_0 = \mathbb{R}_{++}$: $p^\star = -1/e$, optimal point $x^\star = 1/e$
- $f_0(x) = x^3 - 3x,\ \mathbf{dom} f_0 = \mathbb{R}$: $p^\star = -\infty$, local optimum at $x = 1$

Convex Optimization Problems

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \;\; i = 1, \ldots, m \\ & a_i^T x = b_i, \;\; i = 1, \ldots, p \end{array}$$

- $f_0, f_i, \;\; i = 1 \ldots, m$ must be convex
- equality constraints must be affine

Equivalent form:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \;\; i = 1, \ldots, m \\ & Ax = b \end{array}$$

Properties of Convex Optimization Problems

the feasible set of convex optimization problem is convex

- minimize a convex objective function over a convex set

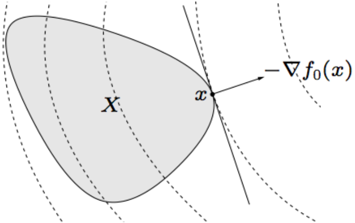
any locally-optimal point is globally optimal

Optimality Conditions For Differentiable Objectives

Suppose the objective f_0 is differentiable and the feasible set is X. Then x is optimal if and only if $x \in X$ and

$$\nabla f_0(x)^T(y - x) \geq 0 \text{ for all } y \in X$$

- $\nabla f_0(x)^T = 0$, or
- $\nabla f_0(x)$ defines a supporting hyperplane to X at x:



Optimality Conditions For Special Cases

- Unconstrained optimization: x is optimal if and only if

$$x \in \mathbf{dom} f_0, \ \nabla f_0(x) = 0$$

- Equality constrained optimization:

$$\text{minimize } f_0(x) \ \text{ subject to } Ax = b$$

x is optimal if and only if there exists ν such that

$$x \in \mathbf{dom} f_0, \ Ax = b, \ \nabla f_0(x) + A^T \nu = 0$$

- Minimization over nonnegative orthant:

$$\text{minimize } f_0(x) \ \text{ subject to } x \succeq 0$$

x is optimal if and only if

$$x \in \mathbf{dom} f_0, \ x \succeq 0, \ x_i \cdot (\nabla f_0(x))_i = 0, \ i = 1, \ldots, n$$

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Importance of Equivalent Problems

a general nonconvex optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) = x_1^2 + x_2^2 \\ \text{subject to} & f_1(x) = x_1/(1 + x_2^2) \leq 0 \\ & h_1(x) = (x_1 + x_2)^2 = 0 \end{array}$$

an equivalent convex problem:

$$\begin{array}{ll} \text{minimize} & f_0(x) = x_1^2 + x_2^2 \\ \text{subject to} & \bar{f}_1(x) = x_1 \leq 0 \\ & \bar{h}_1(x) = x_1 + x_2 = 0 \end{array}$$

operations that produce equivalent problems

- some preserve convexity (useful in reducing complexity)
- some do not preserve convexity (useful in converting nonconvex problems to convex problems)

Change of Variables (Do Not Preserve Convexity)

$\phi : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one mapping

define

$$\bar{f}_i(z) = f_1\left(\phi(z)\right), \;\; i = 0, \ldots, m$$
$$\bar{h}_i(z) = h_i\left(\phi(z)\right), \;\; i = 1, \ldots, p$$

equivalent problem with change of variable $x = \phi(z)$:

$$\begin{array}{ll} \text{minimize} & \bar{f}_0(z) \\ \text{subject to} & \bar{f}_i(z) \leq 0, \;\; i = 1, \ldots, m \\ & \bar{h}_i(z) = 0, \;\; i = 1, \ldots, p \end{array}$$

over variables z

Transformation of Functions (Do Not Preserve Convexity)

$\psi_0 : \mathbb{R} \to \mathbb{R}^n$ is increasing
$\psi_1, \ldots, \psi_m : \mathbb{R} \to \mathbb{R}^n$ satisfy $\psi_i(u) \leq 0$ if and only if $u \leq 0$
$\psi_{m+1}, \ldots, \psi_{m+p} : \mathbb{R} \to \mathbb{R}^n$ satisfy $\psi_i(u) = 0$ if and only if $u = 0$

define

$$\bar{f}_i(x) = \psi_i\left(f_i(x)\right), \;\; i = 0, \ldots, m$$
$$\bar{h}_i(x) = \psi_{m+i}\left(h_i(x)\right), \;\; i = 1, \ldots, p$$

equivalent problem with transformation of functions:

$$\begin{aligned} \text{minimize}_x \quad & \bar{f}_0(x) \\ \text{subject to} \quad & \bar{f}_i(x) \leq 0, \;\; i = 1, \ldots, m \\ & \bar{h}_i(x) = 0, \;\; i = 1, \ldots, p \end{aligned}$$

Slack Variables (Preserve Convexity)

slack variables $s_i \geq 0, \ i = 1, \ldots,$

equivalent problem with slack variables:

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & s_i \geq 0, \ i = 1, \ldots, m \\
& f_i(x) + s_i = 0, \ i = 1, \ldots, m \\
& h_i(z) = 0, \ i = 1, \ldots, p
\end{array}
$$

over variables $x \in \mathbb{R}^n$ and $s \in \mathbb{R}^m$

eliminating (complicated) inequality constraints by adding variables

Eliminating Equality Constraints (Preserve Convexity)

$F \in \mathbb{R}^{n\times k}$ such that $\mathcal{R}(F) = \mathcal{N}(A)$ ($k \geq n - \text{rank}A$)
x_0 such that $Ax_0 = b$

any x s.t. $Ax = b$ can be written as $x = Fz + x_0$, where $z \in \mathbb{R}^k$

equivalent problem with no equality constraints:

$$\begin{array}{ll} \text{minimize} & f_0(Fz + x_0) \\ \text{subject to} & f_i(Fz + x_0) \leq 0, \ \ i = 1, \dots, m \end{array}$$

over variables $z \in \mathbb{R}^k$

eliminating equality constraints and reducing variables by exploiting the structure of $Ax = b$

Introducing Equality Constraints (Preserve Convexity)

original problem of the form:

$$\begin{array}{ll} \text{minimize} & f_0(A_0x + b_0) \\ \text{subject to} & f_i(A_ix + b_i) \leq 0, \ \ i = 1, \dots, m \\ & h_i(x) = 0, \ \ i = 1, \dots, p \end{array}$$

introduce $y_i = A_ix + b_i, \ \ i = 0, \dots, m$

equivalent problem with added equality constraints:

$$\begin{array}{ll} \text{minimize} & f_0(y_0) \\ \text{subject to} & f_i(y_i) \leq 0, \ \ i = 1, \dots, m \\ & y_i = A_ix + b_i, \ \ i = 0, \dots, m \\ & h_i(x) = 0, \ \ i = 1, \dots, p \end{array}$$

over variables x and $y_0, \dots, y_m$

independent objective functions and inequality constraints

Optimizing Over Some Variables (Preserve Convexity)

original problem of the form

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & f_i(x_1) \leq 0, \ i = 1, \ldots, m_1 \\ & \bar{f}_i(x_2) \leq 0, \ i = 1, \ldots, m_2 \end{array}$$

over variables (x_1, x_2)

define $\hat{f}_0(x_1) = \inf \left\{ f_0(x_1, z) | \bar{f}_i(z) \leq 0, \ i = 1, \ldots, m_2 \right\}$

equivalent problem:

$$\begin{array}{ll} \text{minimize} & \bar{f}_0(x_1) \\ \text{subject to} & f_i(x_1) \leq 0, \ i = 1, \ldots, m_1 \end{array}$$

reduce variables (especially if $\inf_z f_0(x_1, z)$ is easy to solve)

popular now for distributed implementation (e.g., ADMM)

Epigraph Form (Preserve Convexity)

epigraph form:

$$\begin{aligned} \text{minimize} \quad & t \\ \text{subject to} \quad & f_0(x) - t \leq 0 \\ & f_i(x_i) \leq 0, \ \ i = 1, \ldots, m \\ & h_i(x) = 0, \ \ i = 1, \ldots, p \end{aligned}$$

over variables x and $t \in \mathbb{R}$

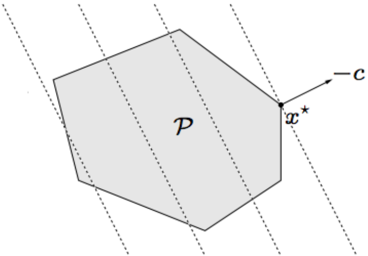
nice (i.e., linear) objective function

Outline

Linear Optimization Problem / Linear Program (LP)

$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

with variable $x \in \mathbb{R}^n$, problem data $G \in \mathbb{R}^{m \times n}$, $A \in \mathbb{R}^{p \times n}$



the "simplest" convex optimization problems

Examples of Linear Programs

diet problem: choose quantities $x_1, \ldots, x_n$ of n foods

- healthy diet requires nutrient i in quantity at least b_i
- one unit of food j contains amount a_{ij} of nutrient i, costs c_j

find the cheapest diet that satisfies nutritional requirements:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

Examples of Linear Programs

Chebyshev center of a polyhedron:

- polyhedron: $\mathcal{P} = \{x \mid a_i^T x \le b_i,\ i = 1, \ldots, m\}$
- represent the largest ball lies in $\mathcal{P}$ as $\mathcal{B} = \{x_c + u \mid \|u\|_2 \le r\}$

observe that ball $\mathcal{B}$ lies in halfspace $a_i^T x \le b_i$

$$\Leftrightarrow \sup \left\{ a_i^T (x_c + u) | \|u\|_2 \le r \right\} = a_i^T x_c + r\|a_i\|_2$$

find the Chebyshev center x_c:

$$\begin{array}{ll} \text{maximize} & r \\ \text{subject to} & a_i^T x_c + \|a_i\|_2 r \le b_i,\ i = 1, \ldots, m \end{array}$$

over variables x_c and r.

Examples of Linear Programs

piecewise-linear minimization: minimize

$$f(x) = \max_{i=1,\ldots,m} (a_i^T x + b_i)$$

first write the epigraph form

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \max_{i=1,\ldots,m} (a_i^T x + b_i) \leq t \end{array}$$

then write the equivalent LP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & a_i^T x + b_i \leq t, \;\; i = 1, \ldots, m \end{array}$$

Linear-Fractional Programming

minimize a ratio of affine functions over a polyhedron:

$$\begin{aligned} \text{minimize} \quad & f_0(x) = \frac{c^T x + d}{e^T x + f} \\ \text{subject to} \quad & Gx \leq h \\ & Ax = b \end{aligned}$$

where $\mathbf{dom} f_0 = \{x \mid e^T x + f > 0\}$

change of variables: $y = \frac{x}{e^T x + f}$, $z = \frac{1}{e^T x + f}$

the equivalent LP

$$\begin{aligned} \text{minimize} \quad & c^T y + dz \\ \text{subject to} \quad & Gy - hz \leq 0 \\ & Ay - bz = 0 \\ & e^T y + fz = 1 \\ & z > 0 \end{aligned}$$

Quadratic Optimization Problems

quadratic program (QP):

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

where $P \in \mathbb{S}^n_+$, $G \in \mathbb{R}^{m \times n}$, $A \in \mathbb{R}^{p \times n}$

quadratically constrained quadratic program (QCQP):

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \;\; i = 1, \ldots, m \\ & Ax = b \end{array}$$

where $P_i \in \mathbb{S}^n_+$, $i = 0, 1, \ldots, m$

Examples of Quadratic Programs

(constrained) regression/least-square:

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 \\ \text{subject to} & l_i \leq x_i \leq u_i, i = 1, \ldots, m \end{array}$$

equivalent QP:

$$\begin{array}{ll} \text{minimize} & x^T(A^TA)x - 2(A^Tb)^Tx + b^Tb \\ \text{subject to} & l_i \leq x_i \leq u_i, i = 1, \ldots, m \end{array}$$

Examples of Quadratic Programs

distance between polyhedra:

- polyhedra $\mathcal{P}_1 = \{x | A_1 x \leq b_1\}$ and $\mathcal{P}_2 = \{x | A_2 x \leq b_2\}$
- distance $\inf \{\|x_1 - x_2\|_2 \mid x_1 \in \mathcal{P}_1,\ x_2 \in \mathcal{P}_2\}$

equivalent QP:

$$\begin{array}{ll} \text{minimize} & \|x_1 - x_2\|_2^2 \\ \text{subject to} & A_1 x_1 \leq b_1 \\ & A_2 x_2 \leq b_2 \end{array}$$

Examples of Quadratic Programs

linear program with random cost

- cost function (vector) $c \in \mathbb{R}^n$ is random
- mean value $\mathbb{E}c = \bar{c}$, variance $\mathbb{E}(c - \bar{c})(c - \bar{c})^T = \Sigma$
- mean cost $\mathbb{E}c^T x = \bar{c}^T x$
- variance of cost $\mathrm{var}(c^T x) = \mathbb{E}\left(c^T x - \mathbb{E}c^T x\right)^2 = x^T \Sigma x$
- minimize the risk-sensitive cost $\mathbb{E}c^T x + \gamma \mathrm{var}(c^T x)$

equivalent QP:

$$\begin{array}{ll} \text{minimize} & \bar{c}^T x + \gamma x^T \Sigma x \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

Second-Order Cone Programming

second-order cone program (SOCP):

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \;\; i = 1, \ldots, m \\ & Fx = g \end{array}$$

where $A_i \in \mathbb{R}^{n_i \times n}$, $F \in \mathbb{R}^{p \times n}$

reduce to LP when $A_i = 0$

reduce to QCQP when $c_i = 0$

Examples of Second-Order Cone Programming

robust linear programming:

- a linear program with uncertainty in a_i:

$$\text{minimize } c^T x \quad \text{subject to } a_i^T x \leq b_i, \ \ i = 1, \ldots, m$$

- a_i in a ellipsoid $a_i \in \mathcal{E}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \leq 1\}$
- robust linear program:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \text{ for all } a_i \in \mathcal{E}_i, \ \ i = 1, \ldots, m \end{array}$$

equivalent SOCP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \|P_i^T x\|_2 \leq b_i, \ \ i = 1, \ldots, m \end{array}$$

Geometric Programming

monomial function with domain $\mathbb{R}^n_{++}$:

$$cx_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$$

where $c > 0$, $a_i \in \mathbb{R}$

posynomial function with domain $\mathbb{R}^n_{++}$:

$$\sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$$

where $c_i > 0$, $a_{ik} \in \mathbb{R}$

geometric programming:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 1, \; i = 1, \dots, m \\ & h_i(x) = 1, \; i = 1, \dots, p \end{array}$$

where $f_0, \dots, f_m$ are posynomials, $h_1, \dots, h_p$ are monomials

Convex Reformulation of Geometric Programming

change of variables: $y_i = \log x_i$

monomials become

$$c(e^{y_1})^{a_1} \cdots (e^{y_n})^{a_n} = e^{a^T y + b}$$

posynomials become:

$$\sum_{k=1}^{K} e^{a_k^T y + b_k}$$

transformation of functions – taking logarithm

$$\begin{array}{ll} \text{minimize} & \log \left(\sum_{k=1}^{K} e^{a_{0k}^T y + b_{0k}} \right) \\ \text{subject to} & \log \left(\sum_{k=1}^{K} e^{a_{0k}^T y + b_{0k}} \right) \le 0, \ \ i = 1, \ldots, m \\ & a_{m+i}^T y + b_{m_i} = 0, \ \ i = 1, \ldots, p \end{array}$$

Outline

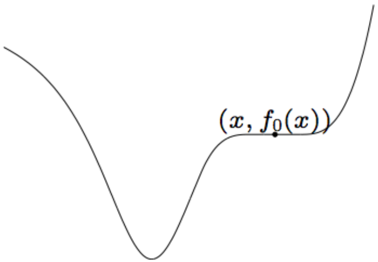
Quasiconvex Optimization Problems

quasiconvex optimization problem:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \ i = 1, \ldots, m \\ & Ax = b \end{array}$$

where f_0 is quasiconvex, $f_1, \ldots, f_m$ are convex

locally optimal points may not be globally optimal



Convex Representation of Sublevel Sets

for any f_0 quasiconvex, there exists a family of convex functions $\{\phi_t\}_{t\in\mathbb{R}}$ such that

$$f(x) \le t \Leftrightarrow \phi_t(x) \le 0$$

t-sublevel set of f_0 is 0-sublevel set of ϕ_t

existence: $\phi_t(x) = \begin{cases} 0 & f(x) \le t \\ \infty & \text{otherwise} \end{cases}$

good for any f, but useless in practice

useful example:

- $f(x) = p(x)/q(x)$ where p is convex and q is concave
- $\phi_t(x) = p(x) - tq(x)$

Quasiconvex Optimization as Convex Feasibility Problems

given t, a feasibility problem:

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & \phi_t(x) \leq 0 \\ & f_i(x) \leq 0, \; i = 1, \dots, m \\ & Ax = b \end{array}$$

- feasible $\Rightarrow p^\star \leq t$
- infeasible $\Rightarrow t \leq p^\star$

use bisection method to find $t = p^\star$