Convex Optimization

Lecture 2 - Convex Sets

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Fall 2017

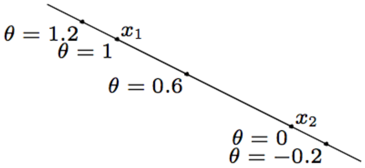
Today's Lecture

Outline

Affine Sets

line through x_1 and x_2: all the points x such that

$$x = \theta x_1 + (1-\theta)x_2,\ \theta \in \mathbb{R}$$



affine set: contains the line through any 2 distinct points in the set

Questions

- What is the affine set containing 3 points in 2-D space?
  the whole space (or a line if these 3 points are on a line)
- Is the solution set of linear equations $\{x | Ax = b\}$ affine?
  yes

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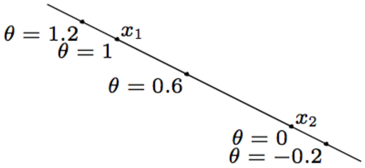

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Convex Sets

line segment between x_1 and x_2: all the points x such that

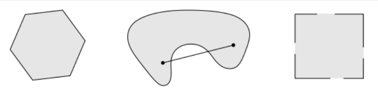
$$x = \theta x_1 + (1-\theta)x_2, \ \theta \in [0,1]$$

convex set: contains line segment between any 2 distinct points in the set

$$x_1, x_2 \in C, \theta \in [0,1] \Rightarrow \theta x_1 + (1-\theta)x_2 \in C$$

Questions

- Are the following sets convex?



yes; no; no

- Is convex set affine? Is affine set convex?

no; yes

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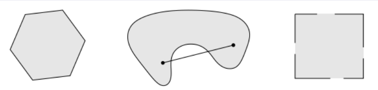


yes; no; no

- Is convex set affine? Is affine set convex?

no; yes

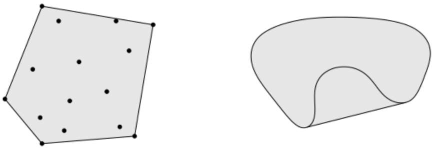
Convex Combination, Convex Hull

convex combination of $x_1, \ldots, x_k$: all points x such that

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

where $\theta_i \geq 0, \theta_1 + \cdots \theta_k = 1$

convex hull conv(S): set of all convex combinations of points in S

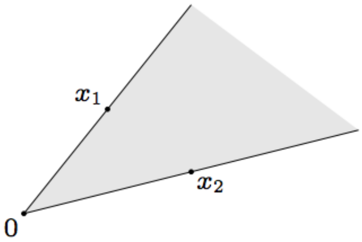


Convex Cone

conic combination of x_1 and x_2: all points x such that

$$x = \theta_1 x_1 + \theta_2 x_2$$

where $\theta_1 \geq 0, \theta_2 \geq 0$



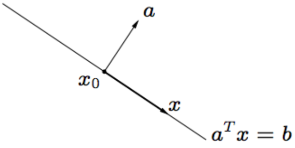
convex cone: set that contains all conic combinations of points in the set

Outline

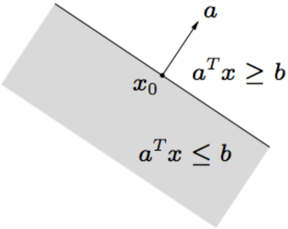
1. Basic Concepts – Affine and Convex Sets
2. Important Examples
3. Operations That Preserve Convexity
4. Separating and Supporting Hyperplanes

Hyperplanes, Halfspaces

hyperplane: set of the form $\{x | a^T x = b\}$, $a \neq 0$



halfspace: set of the form $\{x | a^T x \leq b\}$, $a \neq 0$



Hyperplanes, Halfspaces

Questions

- Is a hyperplane affine? Is it convex?

  yes; yes

- Is a halfspace affine? Is it convex?

  no; yes

Hyperplanes, Halfspaces

Questions

- Is a hyperplane affine? Is it convex?
  yes; yes
- Is a halfspace affine? Is it convex?
  no; yes

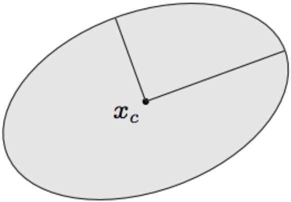
Euclidean Balls, Ellipsoids

Euclidean ball with center x_c and radius r:

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\} = \{x_c + r \cdot u \mid \|u\|_2 \leq 1\}$$

Ellipsoid: set of the form

$$\left\{x \mid (x - x_c)^T P^{-1}(x - x_c) \leq 1\right\},$$ P is symmetric positive definite[1]



[1] positive definite: for all $x \in \mathbb{R}^n$, we have $x^T P x \geq 0$

Norm Balls, Norm Cones

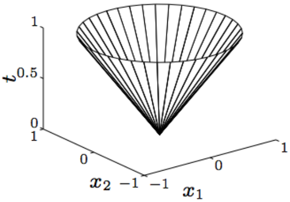
Norm: a function $\|\cdot\|$ that satisfies

- $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$
- $\|tx\| = |t|\|x\|$ for $t \in \mathbb{R}$
- $\|x + y\| \leq \|x\| + \|y\|$

Norm ball with center x_c and radius r:

$$\{x \,|\, \|x - x_c\| \leq r\}$$

Norm cone: set of the form $\{(x, t) \,|\, \|x\| \leq t\}$



Norm Balls, Norm Cones

Questions

- What is a norm ball with 1-norm?

  a square center at the origin and rotated by 45°

Norm Balls, Norm Cones

Questions

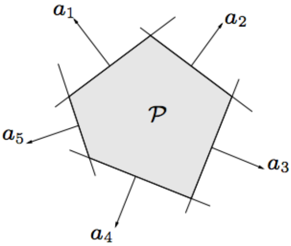
- What is a norm ball with 1-norm?
  a square center at the origin and rotated by 45^o

Polyhedra

Polyhedra: the solution set of finitely many linear inequalities and equalities

$$Ax \leq b, \;\; Cx = d$$

where $A \in \mathbb{R}^{m \times n}$ and $C \in \mathbb{R}^{p \times n}$



intersection of finite number of halfspaces and hyperplanes

Positive Semidefinite Cones

- $\mathbb{S}^n$: set of $n \times n$ symmetric matrices
- $\mathbb{S}^n_+$: symmetric positive semidefinite $n \times n$ matrices

$$X \in \mathbb{S}^n_+ \Leftrightarrow z^T X z \geq 0$$

- $\mathbb{S}^n_{++}$: symmetric positive definite $n \times n$ matrices

$$X \in \mathbb{S}^n_+ \Leftrightarrow z^T X z > 0$$

Questions

- Is $\mathbb{S}^n_+$ convex? Is $\mathbb{S}^n_{++}$ convex?

  yes; yes

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Outline

Operations That Preserve Convexity

How to decide whether a set C is convex?

Method 1: By definition

$$x_1, x_2 \in C,\ \theta \in [0,1] \Rightarrow \theta x_1 + (1-\theta)x_2 \in C$$

Method 2: Show that C is obtained from convex sets (e.g., hyperplanes, norm balls) by operations that preserve convexity

- intersection
- affine functions
- perspective functions
- linear-fractional functions

Intersection

the intersection of (*finite or infinite* number of) convex sets is convex

Questions

- How to prove the intersection rule?
  by definition (exercise on board)
- Prove a polyhedron is convex by the intersection rule?
  intersection of halfspaces and hyperplanes
- Prove positive semidefinite cone $\mathbb{S}_+^n$ is convex by the intersection rule?
  $\mathbb{S}_+^n = \cap_{z \neq 0} \{X \in \mathbb{S}^n | z^T X z \geq 0\}$

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Affine Function

affine function: $f : \mathbb{R}^n \to \mathbb{R}^m$ of the form

$$f(x) = Ax + b,\ A \in \mathbb{R}^{m\times n},\ B \in \mathbb{R}^m$$

- the image of a convex set S under f is convex

$$S \text{ convex} \Rightarrow f(S) = \{f(x)|x \in S\} \text{ convex}$$

- the inverse image of a convex set S under f is convex

$$S \text{ convex} \Rightarrow f^{-1}(S) = \{x|f(x) \in S\} \text{ convex}$$

useful special cases:

- for $S \subseteq \mathbb{R}^n$, $\alpha \in \mathbb{R}$, scaling of S is $\alpha S = \{\alpha x|x \in S\}$
- for $S \subseteq \mathbb{R}^n$, $\alpha \in \mathbb{R}^n$, translation of S is $S + \alpha = \{x + \alpha|x \in S\}$
- for $S \subseteq \mathbb{R}^m \times \mathbb{R}^n$, $\alpha \in \mathbb{R}^n$, projection of S is $T = \{x_1 \in \mathbb{R}^m|(x_1, x_2) \in S, x_2 \in \mathbb{R}^n\}$

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- How to prove the affine rule?
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- Prove a polyhedron is convex by the affine rule?
  $f(x) = (b - Ax, d - Cx)$, polyhedron is $f^{-1}\left(\mathbb{R}_+^m \times \{0\}\right)$
- Is the solution set to linear matrix inequality (LMI)

$$A(x) = x_1A_1 + \cdots x_nA_n \preceq B,\ \ A_i, B \in \mathbb{S}^m$$

  convex? $f(x) = B - A(x)$, the set is $f^{-1}\left(\mathbb{S}_+^m\right)$
- Is hyperbolic cone

$$\left\{x | x^TPx \le (c^Tx)^2,\ c^Tx \ge 0\right\},\ P \in \mathbb{S}_+^n,\ c \in \mathbb{R}^n$$

  convex? $f(x) = (P^{1/2}x, c^Tx)$, the set is f^{-1} (norm cone)

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Perspective

perspective function: $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$ of the form

$$P(x, t) = x/t, \text{ with domain } \mathbb{R}^n \times \mathbb{R}_{++}$$

- the image of a convex set under P is convex
- the inverse image of a convex set under P is convex

Questions

- How to prove the above rules?
  by definition (see book 2.3.3)

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Linear-fractional function

linear-fractional function: $f : \mathbb{R}^n \to \mathbb{R}^m$ of the form

$$f(x) = \frac{Ax + b}{c^T x + d}, \text{ with domain } x | c^T x + d > 0$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $d \in \mathbb{R}$

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  composition of affine and perspective functions

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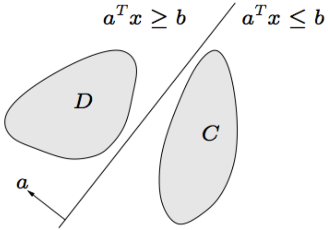
Outline

Separating Hyperplane Theorem

Separating Hyperplane Theorem

if C and D are nonempty disjoint convex sets, there exist $a \neq 0, b$ such that

$$a^T x \leq b \text{ for } x \in C, \text{ and } a^T x \geq b \text{ for } x \in D$$

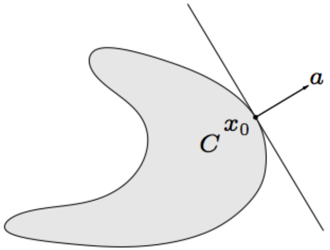


Supporting Hyperplane Theorem

supporting hyperplane to set C at boundary point x_0:

$$\left\{x | a^T x = a^T x_0\right\}$$

where $a \neq 0$, and $a^T x \leq a^T x_0$ for all $x \in C$



the "tangent" line

Supporting Hyperplane Theorem

Supporting Hyperplane Theorem

If C is a convex set, then there exists a supporting hyperplane at every boundary point of C.

Questions

- Prove supporting hyperplane theorem?
  use separating hyperplan theorem on the interior of C and the boundary point x_0

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Questions

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  use separating hyperplan theorem on the interior of C and the boundary point x_0