Convex Optimization

Lecture 14 - Convex Relaxation For Nonconvex Problems

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Today's Lecture

1. Basic Concepts
2. Convex Relaxation
3. Recover Solutions to The Original Problem

Outline

Convex Relaxation For Nonconvex Problems

a lot of practical problems are nonconvex:

- very difficult to solve
- in general, NP-hard to find global optimal solutions

convex relaxation:

- provides bounds on the optimal value
- produces good (not necessarily optimal) solutions
- state-of-the-art performance in many problems!

Nonconvex QCQPs

nonconvex QCQPs:

$$\begin{array}{ll} \text{minimize} & x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} & x^T P_i x + q_i^T x + r_i \leq 0, \;\; i = 1, \ldots, m \end{array}$$

- $P_0, \ldots, P_m$ may not be positive semidefinite
- includes equality constraints (two opposing inequalities)

nonconvex QCQPs include a wide variety of problems

Example – Boolean Least Squares

boolean least squares:

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 \\ \text{subject to} & x_i \in \{-1, 1\}, \;\; i = 1, \ldots, n \end{array}$$

equivalent formulation:

$$\begin{array}{ll} \text{minimize} & x^T A^T A x - 2b^T A x + b^T b \\ \text{subject to} & x_i^2 - 1 = 0, \;\; i = 1, \ldots, n \end{array}$$

maximum likelihood estimation of digital signals

Example – Minimum Cardinality Problems

minimum cardinality problems:

$$\begin{array}{ll} \text{minimize} & \mathbf{card}(x) \\ \text{subject to} & Ax \leq b \end{array}$$

where $\mathbf{card}(x)$ is the number of nonzero elements in x

assume the feasible set is included in the ℓ_∞ ball with radius $R > 0$

equivalent formulation:

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T v \\ \text{subject to} & Ax \leq b \\ & -Rv \leq x \leq Rv \\ & v \in \{0,1\}^n \quad (\Leftrightarrow v_i^2 - v_i = 0, \ i = 1, \ldots, n) \end{array}$$

with optimization variables $x, v \in \mathbb{R}^n$

Example – Minimum Cardinality Problems

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with optimization variables $x, v \in \mathbb{R}^n$

Example – Two-Way Partitioning Problems

two-way partitioning problems:

$$\begin{array}{ll} \text{minimize} & x^T W x \\ \text{subject to} & x_i^2 = 1, \;\; i = 1, \ldots, n \end{array}$$

with optimization variable $x \in \mathbb{R}^n$ and problem data $W \in \mathbb{S}^n$

a feasible x corresponds to a partition

$$\{1, \ldots, n\} = \{i \mid x_i = -1\} \cup \{i \mid x_i = 1\}$$

coefficients in W correspond to costs

- W_{ij}: cost of i and j in the same partition
- $-W_{ij}$: cost of i and j in different partitions

Example – Max-Cut Problem

a graph with n nodes and edges with weights a_{ij}

max-cut problem:

- “cut” the graph (i.e., partition the nodes into two subsets)
- maximize the total weights of edges linking two subsets

applications in circuit design, etc.

Example – Max-Cut Problem

problem formulation:

- a cut $x = \{-1, 1\}^n$
- total weights

$$\frac{1}{2} \sum_{i,j \in \{(i,j) \mid x_i x_j = -1\}} a_{ij} = \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(1 - x_i x_j)$$

a special case of two-way partitioning problem:

$$\begin{array}{ll} \text{minimize} & x^T W x \\ \text{subject to} & x_i^2 = 1, \; i = 1, \ldots, n \end{array}$$

with $W \in \mathbb{S}^n$ where

$$W_{ij} = \begin{cases} \sum_{j=1}^{n} a_{ij} & i = j \\ -a_{ij} & i \neq j \end{cases}$$

Example – Polynomial Optimization

polynomial optimization:

$$\begin{array}{ll} \text{minimize} & p_0(x) \\ \text{subject to} & p_i(x) \le 0, \ \ i = 1, \dots, m \end{array}$$

- $x \in \mathbb{R}^n$: optimization variables
- $p_0, \dots, p_m$ arbitrary polynomials

very general

- include quadratic functions as special cases
- include many combinatorial problems as special cases

$$x_i \in \{0, 1, 2, 3\} \ \Leftrightarrow \ x_i(x_i - 1)(x_i - 2)(x_i - 3) = 0$$

Example – Polynomial Optimization

any polynomial optimization is equivalent to a nonconvex QCQP!

basic idea: adding more variables

example:

$$\begin{array}{ll} \text{minimize} & x^3 - 2xyz + y + 2 \\ \text{subject to} & x^2 + y^2 + z^2 - 1 = 0 \end{array}$$

defining $u = x^2$ and $v = yz$, we have

$$\begin{array}{ll} \text{minimize} & xu - 2xv + y + 2 \\ \text{subject to} & x^2 + y^2 + z^2 - 1 = 0 \\ & u - x^2 = 0 \\ & v - yz = 0 \end{array}$$

with variables u, v, x, y, z

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3. Recover Solutions to The Original Problem

Convex Relaxation – Semidefinite Relaxation

original QCQP:

$$\begin{array}{ll} \text{minimize} & x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} & x^T P_i x + q_i^T x + r_i \leq 0, \;\; i = 1, \ldots, m \end{array}$$

defining $X = xx^T$ and using $x^T P x = \mathbf{tr}(P(xx^T))$, we have

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(P_0 X) + q_0^T x + r_0 \\ \text{subject to} & \mathbf{tr}(P_i X) + q_i^T x + r_i \leq 0, \;\; i = 1, \ldots, m \\ & X = xx^T \end{array}$$

Convex Relaxation – Semidefinite Relaxation

relaxation:

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(P_0 X) + q_0^T x + r_0 \\ \text{subject to} & \mathbf{tr}(P_i X) + q_i^T x + r_i \leq 0, \;\; i = 1, \ldots, m \\ & X \succeq xx^T \end{array}$$

using Schur complement, we have semidefinite relaxation:

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(P_0 X) + q_0^T x + r_0 \\ \text{subject to} & \mathbf{tr}(P_i X) + q_i^T x + r_i \leq 0, \;\; i = 1, \ldots, m \\ & \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \geq 0 \end{array}$$

optimal value of the above SDP provides a lower bound

Convex Relaxation – Lagrangian Relaxation

Lagrangian relaxation

- solve the dual problem
- the optimal value provides a lower bound

dual problem:

$$
\begin{array}{ll}
\text{maximize} & \gamma + \sum_{i=1}^{m} \lambda_i r_i + r_0 \\
\text{subject to} & \begin{bmatrix} (P_0 + \sum_{i=1}^{m} \lambda_i P_i) & (q_0 + \sum_{i=1}^{m} \lambda_i q_i)/2 \\ (q_0 + \sum_{i=1}^{m} \lambda_i q_i)^T/2 & -\gamma \end{bmatrix} \succeq 0 \\
& \lambda_i \geq 0, \;\; i = 1, \ldots, m
\end{array}
$$

with optimization variable $\lambda \in \mathbb{R}^m$

semidefinite and Lagrangian relaxations are duals of each other

Example – Convex Relaxation of Boolean Least Squares

boolean least squares:

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 \\ \text{subject to} & x_i \in \{-1, 1\}, \ i = 1, \ldots, n \end{array}$$

semidefinite relaxation:

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(A^T A X) - 2b^T A x + b^T b \\ \text{subject to} & \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \succeq 0 \\ & X_{ii} = 1, \ i = 1, \ldots, n \end{array}$$

Outline

Recover Solutions to The Original Problem

one simple heuristic: rounding

semidefinite relaxation of boolean least squares:

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(A^T A X) - 2b^T A x + b^T b \\ \text{subject to} & \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \succeq 0 \\ & X_{ii} = 1, \; i = 1, \ldots, n \end{array}$$

with optimal solution $X_{\text{sdr}}, x_{\text{sdr}}$

an approximate solution x^* to the original problem

$$x_i^* = \begin{cases} +1 & x_{\text{sdr},i} \geq 0 \\ -1 & x_{\text{sdr},i} < 0 \end{cases}$$

one approximate solution, no performance guarantee

Recover Solutions by Randomization

semidefinite relaxation of nonconvex QCQP:

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(P_0 X) + q_0^T x + r_0 \\ \text{subject to} & \mathbf{tr}(P_i X) + q_i^T x + r_i \leq 0, \;\; i = 1, \ldots, m \\ & \begin{bmatrix} X & x \\ x^T & 1 \end{bmatrix} \succeq 0 \end{array}$$

with optimal solution $X_{\text{sdr}}, x_{\text{sdr}}$

x is a Gaussian random variable with $x \sim \mathcal{N}\left(x_{\text{sdr}}, X_{\text{sdr}} - x_{\text{sdr}} x_{\text{sdr}}^T\right)$

$$x_i^* = \begin{cases} +1 & x_{\text{sdr},i} \geq 0 \\ -1 & x_{\text{sdr},i} < 0 \end{cases}$$

x solve the nonconvex QCQP "on average"

$$\begin{array}{ll} \text{minimize} & \mathbf{E}\left(x^T P_0 x + q_0^T x + r_0\right) \\ \text{subject to} & \mathbf{E}\left(x^T P_i x + q_i^T x + r_i\right) \leq 0, \;\; i = 1, \ldots, m \end{array}$$

Recover Solutions by Randomization

recovery by randomization:

- get sufficiently many samples $x \sim \mathcal{N}\left(x_{\text{sdr}}, X_{\text{sdr}} - x_{\text{sdr}}x_{\text{sdr}}^T\right)$
- get a feasible solution from each sample (e.g., through rounding)
- pick the best feasible solution

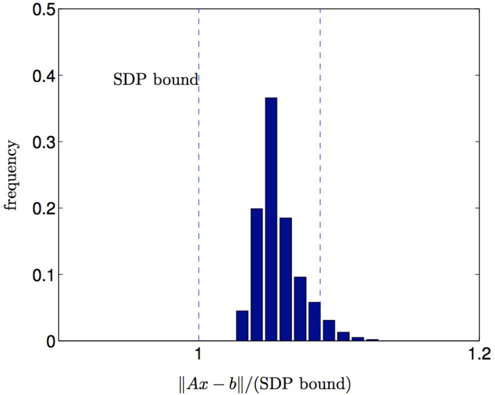
sufficiently many approximate solution to choose from

performance guarantee in many cases

$$p_{\text{sdr}}^\star \leq p^\star \leq \alpha p_{\text{sdr}}^\star \text{ with } \alpha > 1$$

Examples – Boolean Least Squares

boolean least squares with $A \in \mathbb{R}^{150 \times 100}$



more about maximum likelihood detection:
Luo *et al*, "Semidefinite Relaxation of Quadratic Optimization Problems," *IEEE Signal Processing Magazine*, 2010.